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Stochastic Dynamics and Bifurcation Behavior of Nonlinear Nonconservative Systems in the Presence of Noise

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The main objectives of the completed work are to develop mathematical techniques to reduce the dimensionality of multidegree-of-freedom nonlinear systems near bifurcation points and to solve for the response statistics of the reduced system. The asymptotic behavior of nonlinear dynamical systems in the presence of noise is studied using the method of stochastic normal forms. The crucial point in the normal form computations is to find the resonant terms that cannot be eliminated through a nonlinear change of variables. Subsequent to reduction of the dimensionality, a Markovian approximation is used to obtain the associated stochastic normal forms. The key result is that the second order stochastic terms have to be retained in the normal form computations in order to capture the contributions of the stable modes stochastic components to the critical modes drift terms. It is also shown that the method of extended stochastic averaging is in fact "equivalent" to stochastic normal forms for a specific class of nonlinear systems. In addition, mean square stability of the response is obtained and the bifurcation behavior

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19. Abstract

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Abstract

The main objectives of the completed work are to develop mathematical techniques to reduce the dimensionality of multidegree-of-freedom nonlinear systems near bifurcation points and to solve for the response statistics of the reduced system. The asymptotic behavior of nonlinear dynamical systems in the presence of noise is studied using the method of stochastic normal forms. The crucial point in the normal form computations is to find the resonant terms that cannot be eliminated through a nonlinear change of variables. Subsequent to reduction of the dimensionality, a Markovian approximation is used to obtain the associated stochastic normal forms. The key result is that the second order stochastic terms have to be retained in the normal form computations in order to capture the contributions of the stable modes stochastic components to the critical modes drift terms. It is also shown that the method of extended stochastic averaging is in fact "equivalent" to stochastic normal forms for a specific class of nonlinear systems. In addition, mean square stability of the response is obtained and the bifurcation behavior and associated stationary and transient probability density functions for the reduced stochastic system are determined. Finally, the general results are applied to the study of the dynamics of aircraft at high angles of attack, plates under gas flow, structures under follower forces, and propellant lines conveying pulsating fluid.

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1. Introduction

The goal of this work is to obtain results pertaining to the statistical as well as the sample behavior of nonlinear structures subjected to random excitations. Among many results obtained in this work for nonlinear stochastic systems, the results pertaining to the problem of stabilization by noise are of practical significance. These effects have been demonstrated by the P.I. for both gyroscopic and nonconservative systems [1,2].

In order to understand the bifurcation behavior of a dynamical system, a reduction in the mathematical complexity of the n-dimensional problem is required. Often certain variables which are asymptotically stable can be eliminated as being unimportant with the essential behavior of the system restricted to the dynamics of the remaining critical variables. The initial work involves developing mathematical techniques such as stochastic normal forms [3] and extended methods of stochastic averaging [4] to approximate multidegree of freedom nonlinear structures subjected to random excitations by lower dimensional Markov diffusive process. Thus, as a first step we have developed and extended mathematical techniques, to reduce the dimensionality of nonlinear stochastic systems near bifurcation points. Subsequent research involves applying these techniques and proposed methods (see, for example, [5]) in order to obtain an understanding of co-dimension one [6] and co-dimension two [7] stochastic bifurcations. The major goals of this research effort are to examine the stochastic dynamics, stability and bifurcation behavior of various nonlinear stochastic problems with direct impact on the mission of AFOSR. Such problems include: aircraft at high angles of attack under the effect of atmospheric turbulence; rotating shafts and rotating systems under pulsating loads; propellant lines conveying turbulent flow, etc.

The PI has completed most of his objectives and the results from this project have revealed new features in the theory of nonlinear stochastic dynamics. Highlights of these features are briefly discussed in the following subsections.

2. Development of Mathematical Techniques

When a multidegree-of-freedom mechanical system undergoes a bifurcation, it does so only in a few degrees of freedom. The simplest deterministic example to point out is when a single mode becomes unstable due to a control parameter μ being slightly increased beyond a critical value μ_c. For example, μ and μ_c represent the axial and Euler loads respectively, in buckling of a column. In the vicinity of μ_c, the temporal evolution of the motion of the critical mode in the first approximation is given by $\dot{x} = (\mu - \mu_c)\, x + ax^3$. This situation becomes more complicated when a set of control parameters $\underset{\sim}{\mu}$ are varied in such a way that several modes may simultaneously become marginally unstable. In such situations, the system is said to undergo a multiple bifurcation. The associated simplest possible amplitude equations which capture the complete dynamics of the original system in the vicinity of $\underset{\sim}{\mu_c}$ are called the normal form.

For deterministic systems, in addition to the theory of normal forms, the theory of center manifolds and method of averaging are particularly useful in reducing the dimensionality of large nonlinear dynamical systems. However, consistent methods for the analysis of multidegree-of-freedom stochastic nonlinear systems are currently lacking in the engineering community. The mathematical techniques presented in this section, namely stochastic normal forms and extended stochastic averaging, respond to this need. In order to understand the complex

interaction between noise and the inherent nonlinearities in mechanical systems and their bifurcation behavior, a reduction in the mathematical complexity of the n-dimensional problem is required, as discussed, in which the dynamics of the response are captured in the remaining critical modes. Often certain variables which are asymptotically stable can be eliminated as being unimportant with the essential behavior of the system restricted to the dynamics of the remaining critical modes. To this end, the method of averaging was extended, following Papanicolaou and Kohler [8], by the authors [4] to include the analysis of nonlinear systems which exhibit co-dimension one bifurcations. Application of this method to study stochastically perturbed general nonconservative problems was presented by Sri Namachchivaya and Tien [2].

The ideas of center manifold and normal forms were extended to stochastic systems by Knobloch and Wissenfeld [9] and Coulett [10], respectively. The applicability of the method of normal forms to nonlinear stochastic systems was demonstrated by Sri Namachchivaya and Hilton [11]. However, these extensions were unsuccessful in capturing the contributions of the stochastic components of the stable modes to the critical mode drift terms. Such effects were shown to exist by Sri Namachchivaya and Lin [4] using extended stochastic averaging. The goal of this work is to present the method of stochastic normal forms developed by Sri Namachchivaya and Leng [3], in order to reduce the the dimensionality of nonlinear stochastic systems near bifurcation points. Furthermore, it has been shown that these two methods are, in fact, equivalent for a specific class of nonlinear systems. We also wish to add that an alternate approach has been used by Caughey [12] to analyze nonlinear stochastic systems. This approach replaces a nonlinear stochastic system without an exact

solution by an "equivalent" system with an exact solution chosen in some optimal fashion. There need not be any reduction in dimension. The method of stochastic normal forms differs because it replaces the original system by an "equivalent" system of *lower* dimension by the elimination of stable modes.

There are two approaches to obtaining normal forms in deterministic systems. As in Guckenheimer and Holmes [13], in the first method, one first computes the lower dimensional center manifold on which the dynamics reduces for large times and then a nonlinear change of coordinates is applied to transform a small dimensional system to normal form. In the second method, one systematically expands the original vector fields in powers of the amplitudes of critical modes, as in Elphick et al. [14], to yield both the normal form and center manifold. This paper outlines a method which has its basis in [3,14].

2.1 Method of Stochastic Normal Forms

The theory of normal forms goes back to as early as Euler; Poincare [15] and Birkoff [16] contributed a more definite form of the theory. Poincare [15] considered the problem of reducing a system of differential equations of the form

$$\frac{dx}{dt} = Ax + f(x) \quad \text{to} \quad \frac{dy}{dt} = Ay\,, \quad x \,\varepsilon\, R^n\,, \quad y \,\varepsilon\, R^n\,. \tag{1}$$

The formal solution of this problem deals with finding near-identity coordinate transformations $x = y + \Phi(y)$ which eliminate the analytic expressions of the nonlinear terms. It was shown that such a formal

solution exists provided the above system is hyperbolic and the eigenvalues λ_j of the diagonalizable matrix A satisfy

$$\lambda_i \neq \sum k_l \lambda_l \quad \text{for} \quad j = 1,2,\ldots,n, \quad |k| = \sum k_l \geq 2 \tag{2}$$

where k is an integer vector $k = (k_1, k_2, \ldots, k_n)$ with $k_l \geq 0$. Furthermore, it was proven that if, in addition to the above results, the eigenvalues lie strictly to one side of a line through the origin in the complex plane, then the formal series $\Phi(y)$ is convergent. If the system is nonhyperbolic or the condition (2) is violated, the analytic expressions of the nonlinear terms cannot be completely eliminated. The normal forms of equation (1) are dictated by the nature of the linear operator and contain only resonant nonlinear terms that cannot be eliminated through a nonlinear change of variables. Thus, the nonlinear system in (1) can be reduced to

$$\frac{dy}{dt} = Ay + g(y), \quad y \,\varepsilon\, R^n, \tag{3}$$

where g is simpler than f and the resulting simplified nonlinear equations are said to be in normal form. Such reductions have been widely used to study deterministic autonomous and nonautonomous systems and Arnold [17] contains a good exposure of this subject. In bifurcation problems, the eigenvalues of the linear operator A are composed of two sets, one on the imaginary axis and the other with strictly negative real parts. The linear vector space E associated with A can also be divided accordingly as $E = E_c \oplus E_s$ such that $x_c \,\varepsilon\, E_c$ and $x_s \,\varepsilon\, E_s$ with $x = x_c \oplus x_s$.

The purpose of the theory of normal forms in our investigations is two-fold: first, to extend the normal form theory to incorporate nonlinear stochastic systems and secondly, to demonstrate the relationship between stochastic averaging and normal form theory for non-nilpotent systems. To this end, consider a dynamical system governed by nonautonomous differential equations in R^n

$$\dot{x} = A(\eta)x + f(x , \eta) + \sigma\xi(t)\, B(\eta)x = A(\eta)x + f(x , \eta) + \sigma F(x, t, \eta) \tag{4}$$

which depend on two external parameters η and σ. The matrices A and B are nxn matrices which depend smoothly on η, $\xi(t)$ is a stationary stochastic process with zero mean representing the parametric excitations, and $x = 0$ is the trivial solution of Eq. (4) for all values of η and σ. The nonlinear function f is a vector function which is smooth in its arguments and the i^{th} component of the r^{th} order polynomial can be represented by

$$f_i^r(x,0) = \sum_m f_{i,m}^r x^m = \sum_m f_{i,m_1,m_2\dots m_n}^r x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}\,,\ f_i = \sum_{r=1}^{N} f_i^r \tag{5}$$

where $m = (m_1, m_2 ,\dots, m_n)$ are non-negative integers, $x^m = x_1^{m_1} \cdots x_2^{m_2} \cdots x_n^{m_n}$ is an r^{th} order monomial such that $\Sigma\, m_j = r$, $f_{i,m}^r$ are the coefficients of the monomial with a particular combination of $(m_1, m_2 ,\dots, m_n)$ and the summation is over all such monomials. Furthermore, the trivial solution of Eq. (4) in the absence of stochastic excitation, i.e., $\sigma = 0$, loses stability and undergoes a co-dimension one bifurcation, namely, either a Hopf or a simple bifurcation at η=0, and the associated linear operator for these cases is $A(0) = \mathrm{diag}\,\{i\omega_1 , - i\omega_1 , \lambda_3 ,\dots, \lambda_n\}$, or $\mathrm{diag}\,\{0, \lambda_2 , \lambda_3 ,\dots, \lambda_n\}$.

Consider a near-identity nonlinear transformation

$$x = y + W(y) + \sigma U(y,t) + \sigma^2 V(y,t)$$

where W(y) is a homogeneous vector polynomial of degree k, k being the lowest order nonlinearity that exists in Eq. (4), U(y,t) and V(y,t) are vector polynomials with time dependent coefficient. Interpreting Eq. (4) in the Stratonovich sense, we want the transformation to yield

$$\dot{y} = A(\eta)\, y + g\,(y,\eta) + \sigma G\,(y,\eta,\xi(t)) + \sigma^2 H\,(y,\eta,\xi(t)) \tag{6}$$

where $g(y,0) = \sum_m g^r_{i;m}\, y^m$, and y^m are the r^{th} order monomials such that $\sum m_j = r$ with $G(y,0,\xi(t))$ and $H(y, 0, \xi(t))$ at least linear in y. Now, we define the Lie bracket of W and Ay as

$$L_A = [W\,,\,Ay] = \frac{\partial W}{\partial y} \cdot Ay - AW\ . \tag{7}$$

Considering now a monomial in the i^{th} component of W, using the fact that A is diagonal and the notation of Eq. (5), the above equation yields

$$L_A\, W^k_{i;m}\, y^m = \left[(m,\lambda) - \lambda_i\right] W^k_{i;m}\, y^m\ . \tag{8}$$

Equating the monomials of order k, we can then evaluate the coefficients of the monomial elements of W by solving

$$L_A\, W^k_{i;m} = \tilde{f}^k_{i,m} - \tilde{g}^k_{i;m} - g^k_{i;m} \equiv h^k_{i;m}\ , \quad \Sigma\, m_j = k \tag{9}$$

where $\tilde{f}^k_{i;m}$, $\tilde{g}^k_{i;m}$ are the coefficients of the monomial y^m of the following polynomials of degree k, respectively,

$$\sum_{r=2}^{k} f_i^r\,(y + W(y))\,,\quad \sum_{s=2}^{k-1} \frac{\partial W_i^{(k+1-s)}(y)}{\partial y_j}\, g_j^s(y)\,.$$

L_A is called the homological operator, since $L_A : H_k(R^n) \to H_k(R^n)$, where $H_k(R^n)$ is the space of homogeneous vector polynomials of degree k on R^n. The crucial point in the normal form computations is to find a homogeneous polynomial vector field of degree k in a space complementary to the range of the homological operator. The dimension of the vector space increases with k. This makes the computations cumbersome for large k. Furthermore, since the matrix A is diagonal, the image of L_A, Im(L_A), and its null space, ker (L_A), span the whole space. Consequently, in order to solve Eq. (9) we should have $h_{i,m}^k \in \mathrm{Im}\,(L_A)$ and this gives a condition that $g_{i;m}^k$ may be chosen in the null space of L_A, i.e., $g_{i,m}^k \in \ker\,(L_A)$ when the null space is not empty. Furthermore, from Eq. (8), the resonance condition for the deterministic terms reduces to

$$\sum_{l=1}^{n} m_l\, \lambda_l - \lambda_i = 0 \qquad \text{for all } i = 1,2,\ldots,n\,.$$

Similarly, equating the terms of order k in y and 1 in σ, the time dependent coefficients of the monomial elements of U are evaluated from

$$\left(\frac{\partial}{\partial t} + L_A\right) U_{i;m}^k(t) = \widetilde{F}_{i;m}^k(t) - \widetilde{G}_{i;m}^k(t) - G_{i;m}^k(t) \tag{10}$$

where $\widetilde{F}_{i,m}^k(t)$, $\widetilde{G}_{i,m}^k(t)$ are the coefficients of the monomial y^m of the following polynomials of degree k, respectively,

$$\sum_{r=1}^{k} F_i^r\,(y + W(y),t) + \sum_{s=1}^{k-1} \frac{\partial f_i^{\,k+1-s}(y + W(y))}{\partial x_j}\; U_j^s\,(y,t)\ ,$$

$$\sum_{s=1}^{k-1} \frac{\partial W_i^{\,k+1-s}(y)}{\partial y_i}\; G_j^s\,(y,t) + \sum_{s=2}^{k} \frac{\partial U_i^{\,k+1-s}(y,t)}{\partial y_j}\; g_j^s\,(y)\ .$$

Identifying the terms of order k in y and 2 in σ, the time dependent coefficients of the monomial elements of V are evaluated from

$$\left(\frac{\partial}{\partial t} + L_A\right) V_{i;m}^k(t) = \widehat{F}_{i;m}^k(t) - \widetilde{H}_{i;m}^k(t) - H_{i;m}^k(t) \tag{11}$$

where $\widehat{F}_{i;m}^k(t)\,,\ \widetilde{H}_{i;m}^k(t)$ are the coefficients of the monomial y^m of the following polynomials of degree k, respectively,

$$\sum_{s=1}^{k} \frac{\partial F_i^{(k+1-s)}(y+W(y),t)}{\partial y_i} U_j^s(y,t) + \sum_{s=1}^{k-1} \frac{\partial f_i^{(k+1-s)}(y+W(y))}{\partial y_j} V_j^s(y,t)$$

$$+ \sum_{s=1}^{k-1} \sum_{r=1}^{k-s} \frac{\partial^2 f_i^{(k+2-r-s)}(y)}{\partial y_j \partial y_k} U_j^s(y,t)\, U_k^r(y,t)\ ,$$

$$\sum_{s=1}^{k} \frac{\partial U_i^{(k+1-s)}(y,t)}{\partial y_j}\; G_j^s(y,t) + \sum_{s=1}^{k-1} \frac{\partial W_i^{(k+1-s)}(y)}{\partial y_j} H_j^s(y,t)$$

$$+ \sum_{s=2}^{k} \frac{\partial V_i^{(k+1-s)}(y,t)}{\partial y_j}\, g_j^s(y)$$

In Eqs. (10) and (11), the derivative $(\partial/\partial t)$ acts only on the functions $U^{k}_{i;m}(t)$ and $V^{k}_{i;m}(t)$, respectively. Equation (10) contains the results from Eq. (9), and Eq. (11) contains the results from both Eqs. (9) and (10). In the above expressions, the repeated subscripts imply summation up to n. After taking Fourier transforms, the resonance condition for the stochastic terms becomes

$$j\Omega + \sum_{l=1}^{n} m_l \lambda_l - \lambda_i = 0 \quad \text{for all } i = 1,2,\ldots n . \tag{12}$$

and this expression dictates the stochastic normal form. The coefficients of the monomial elements W, U and V are solved from Eqs. (9), (10) and (11), respectively, keeping in mind both the deterministic and stochastic resonance terms.

The noise terms in the critical modes which contain stable variables can be eliminated. Let the noise term be of the form

$$\left(y_{c_1}^{m_1} \ldots\ldots y_{c_p}^{m_p}\right)\left(y_{s_1}^{n_1} \ldots\ldots y_{s_q}^{n_q}\right) \xi(t)$$

then the stochastic resonance condition is

$$j\Omega + \sum_{l=1}^{p} m_l \lambda_{c_l} + \sum_{l=1}^{q} n_l \lambda_{s_l} - \lambda_{c_i} = 0$$

where the λ_{c_l}'s are either zero or pure imaginary and the λ_{sj}'s have negative real parts. Since at least one of the n_j's is a non-negative integer, it is obvious that the resonance condition can never be satisfied for any

value of Ω. Hence, the noise terms containing the stable variables are removed from the critical modes.

Similarly, in the case of linear multiplicative noise, noise terms of the form $y_{cl}\,\xi(t)$ can be removed from the stable modes. Checking the resonance condition,

$$j\Omega + (1)\lambda_{c_l} - \lambda_{s_i} = 0\ , \quad \Sigma m_l + \Sigma n_l = 1$$

since λ_{c_l} is zero or pure imaginary and λ_{s_i} has a negative real part, the resonance condition cannot be satisfied for any Ω. It is worth pointing out that, for higher order noise terms, such a decoupling may not be possible because of "stochastic resonance". This can be illustrated through an example in which $\lambda_{c_1} = j\omega$, $\lambda_{c_2} = -j\omega$ (Hopf bifurcation), and $\lambda_{s_1} = -\gamma$. Let the stable mode contain a noise term of the form $y_{c_1}^{m_1}\, y_{c_2}^{m_2}\, y_{s_1}\, \xi(t)$. The resonance condition is

$$j\Omega + m_1\,(j\omega) + m_2\,(-j\omega) + 1\,(-\gamma) - (-\gamma) = 0\ , \text{ i.e, } \ j\,(\Omega + (m_1 - m_2)\,\omega) = 0$$

which is satisfied for $\Omega = (m_2 - m_1)\,\omega$. Thus, it is not always possible to remove such terms unless the noise $\xi(t)$ has no energy at the frequencies $\Omega = (m_2 - m_1)\,\omega$.

More specifically, putting $x = \{x_c , x_s\}$, $f = \{f_c , f_s\}$, $W = \{W_c , W_s\}$, $U = \{U_c , U_s\}$, $V = \{V_c , V_s\}$ and $A = \text{diag}\,\{A_c , A_s\}$ where the eigenvalues of A_c are pure imaginary or zero and the eigenvalues of A_s have negative real parts, the normal form procedure can be stated as: given

$$\frac{dx_c}{dt} = A_c\, x_c + f_c\,(x_c , x_s , \eta) + \sigma F_c\big(x_c , x_s , \eta\ ; \xi(t)\big)$$

$$\frac{dx_s}{dt} = A_s\, x_s + f_s\,(x_c , x_s , \eta) + \sigma F_s\big(x_c , x_s , \eta\ ; \xi(t)\big)$$

the near identity transformations

$$x_c = y_c + W_c(y_c, y_s, \eta) + \sigma U_c\left(y_c, y_s, \xi(t), \eta\right) + \sigma^2 V_c\left(y_c, y_s, \xi(t), \eta\right)$$
$$x_s = y_s + W_s(y_c, y_s, \eta) + \sigma U_s\left(y_c, y_s, \xi(t), \eta\right) + \sigma^2 V_s\left(y_c, y_s, \xi(t), \eta\right).$$

yield

$$\frac{dy_c}{dt} = A_c y_c + g_c(y_c, \eta) + \sigma G_c\left(y_c, \xi(t), \eta\right) + \sigma^2 H_c\left(y_c, \xi(t), \eta\right)$$

$$\frac{dy_s}{dt} = A_s y_s + g_s(y_c, y_s, \eta) + \sigma G_s\left(y_c, y_s, \xi(t), \eta\right) + \sigma^2 H_s\left(y_c, y_s, \xi(t), \eta\right)$$

and W, U and V are such that g_c, G_c and H_c are as simple as possible and take values in E_c.

Since the theory of normal forms arises from perturbation analysis, it is usually presented for equations which contain a small parameter. In nonlinear equations, the small parameters can be easily introduced by scaling of the state variables. In order to study the interplay between the deterministic and stochastic components of the system, it is assumed that the nonvanishing nonlinear term in the normal form is of the order ε^2 and $\sigma=\varepsilon$. Furthermore, for simplicity, it is assumed that the vector field f is an odd function of x and the stochastic terms are linear in x and can be partitioned as

$$\begin{Bmatrix} F_c^1\left(x_c, x_s, \xi(t), \eta\right) \\ F_s^1\left(x_c, x_s, \xi(t), \eta\right) \end{Bmatrix} = \begin{bmatrix} B_{cc}(\xi(t), \eta) & B_{cs}(\xi(t), \eta) \\ B_{sc}(\xi(t), \eta) & B_{ss}(\xi(t), \eta) \end{bmatrix} \begin{bmatrix} x_c \\ x_s \end{bmatrix} \tag{13}$$

As shown earlier, the noise terms in the critical modes which contain the stable variables can be eliminated. For the case of linear multiplicative noise, the same goes for the $B_{sc}(t)\, x_c$ terms in the stable mode equations.

Hence, performing the normal form transformation described earlier, we obtain

$$\begin{Bmatrix} \dot{y}_c \\ \dot{y}_s \end{Bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix} \begin{Bmatrix} y_c \\ y_s \end{Bmatrix} + \varepsilon^2 \begin{bmatrix} g_c^3\,(y_c) \\ g_s^3\,(y_c\,,\,y_s) \end{bmatrix} + \varepsilon \begin{bmatrix} B_{cc}(t) & 0 \\ 0 & B_{ss}(t) \end{bmatrix} \begin{Bmatrix} y_c \\ y_s \end{Bmatrix}$$

$$+\,\varepsilon^2 \begin{bmatrix} B_{cs}(t)\;U_{sc}(t) & 0 \\ 0 & B_{sc}(t)\;U_{cs}(t) \end{bmatrix} \begin{Bmatrix} y_c \\ y_s \end{Bmatrix} \tag{14}$$

where

$$\dot{U}_{cs} = A_c\,U_{cs} - U_{cs}\,A_s + B_{cs}(t) \tag{15}$$

$$\dot{U}_{sc} = A_s\,U_{sc} - U_{sc}\,A_c + B_{sc}(t) \tag{16}$$

Using the fact that the noise is linear multiplicative allows one to decouple the stochastic terms in (13) leading to Eqs. (14), (15) and (16), which provides the $0(\varepsilon^2)$ contribution from the stable modes to the critical modes. The method of normal forms has effectively uncoupled the critical modes from the stable ones. It is worth pointing out that the deterministic part of the uncoupled system is the same as the deterministic normal form for the system unperturbed by noise. The nonlinear vector function $g_c(y_c)$ for various co-dimension one and two bifurcations are given in Arnold [17] and Guckenheimer and Holmes [13]. It remains now to solve Eqs. (15) and (16) in order to completely obtain the stochastic components of the normal form.

Before proceeding further, Eq. (14) is brought to a "standard form" by using a transformation

$$y_c = e^{A_c t}\,z$$

and Eq. (14a) can be replaced by a differential equation in z as

$$\frac{dz}{dt} = \varepsilon^2 e^{-A_c t} g_c^3\left(e^{A_c t} z\right) + \varepsilon e^{-A_c t} B_{cc}(t)\, e^{-A_c t} z + \varepsilon^2 e^{-A_c t} B_{cs}(t) V_{sc}(t)\, e^{A_c t} z \tag{17}$$

Using the fact that the normal form $g_c^3(y_c)$ lies in the null space of the homological operator L_A, i.e.,

$$\frac{\partial g_c^3(y_c)}{\partial y_c} A_c y_c - A_c g_c^3(y_c) = 0$$

the total differential of $g_c^3(y_c) = g_c^3(e^{A_c t} z)$ can be written as

$$\frac{d}{dt}\left[g_c^3\left(e^{A_c t} z\right)\right] = \frac{\partial g_c^3}{\partial y_c} A_c e^{A_c t} z = \frac{\partial g_c^3}{\partial y_c} A_c y_c = A_c g_c^3(y_c) \tag{18}$$

Equation (18) is a linear first order ordinary differential equation in $g_c^3(y_c)$, whose solution is

$$g_c^3(y_c) = e^{A_c t} g_c^3(x_c, t{=}0) = e^{A_c t} g(z)$$

Thus Eq. (17) can be rewritten as

$$\frac{dz}{dt} = \varepsilon^2 g_c^3(z) + \varepsilon e^{-A_c t} B_{cc}(t)\, e^{A_c t} z + \varepsilon^2 e^{-A_c t} B_{cs}(t) V_{sc}(t)\, e^{A_c t} z \tag{19}$$

2.2 Markov Approximation of Reduced System

In this section, the above lower dimensional equations are replaced by diffusive Markov processes whose transition probabilities at time intervals Δt ($\Delta t >> \tau_{cor}$) are approximately the same as those of the original processes. There are two ways of deriving the drift and diffusion

coefficients which completely describe the Markov process. In the first method, the increments $x(t_2) - x(t_1)$ and $x(x_4) - x(t_3)$, where $t_1 < t_2 < t_3 < t_4$, are assumed independent when the correlation time of the smooth process is much smaller than the relaxation time of the reduced system (Stratonovich [18]). The second method deals with the asymptotic behavior of the solution of the lower dimensional system when τ_{cor} tends to zero. The use of the first method to calculate the drift and diffusion terms is shown in the Appendix of [3]. A physical interpretation of this approach which is more appealing to engineers is given by Lin [19]. It will be shown that the approximation of the solution of the lower dimensional equations by a Markov process give rise to the same drift and diffusion terms as those obtained in the extended stochastic averaging technique [4]. However, for simplicity, consider only two cases, one in which Eqs. (15) and (16) will be solved explicitly.

<u>Case 1</u>: $A_c = 0$ (divergence instability)

In this case, Eqs. (15) and (16) reduce to:

$$\dot{U}_{cs} = - U_{cs} A_s + B_{cs}(t) \tag{20}$$

$$\dot{U}_{sc} = A_s U_{sc} + B_{sc}(t) \tag{21}$$

The required particular solution for the normal form transformation is

$$U_{cs}(t) = \left(\int_t^{\infty} B_{cs}(s)\, e^{A_s s}\, ds \right) e^{-A_s t} \tag{22}$$

$$U_{sc}(t) = e^{A_s t} \left(\int_{-\infty}^{t} e^{-A_s s}\, B_{sc}(s)\, ds \right) \tag{23}$$

Hence, the reduced system is (where z now represents the variable after transformation)

$$z = \varepsilon^2 F_c(z) + \varepsilon B_{cc}(t) z + \varepsilon^2 \left\{ \left(B_{cs}(t) \int_{-\infty}^{t} e^{-A_s(s-t)} B_{sc}(s)\, ds \right) z \right\} \tag{24}$$

Using the Markovian approximation and performing a time-translation, it is found that the drift contribution from the stable modes to the drift term of the critical mode is

$$M_t \left\{ E \left[\left(\int_{-\infty}^{0} B_{cs}(t)\, e^{-A_s \tau} B_{sc}(t+\tau)\, d\tau \right) x_c \right] \right\} \tag{25}$$

which agrees with that obtained from the extended averaging theorem.

<u>Case 2</u> : $A_c = \mathrm{diag}\,\{-j\omega_1 , j\omega_1 , \ldots\ldots, -j\omega_n , j\omega_n\}$ (flutter instability)

In this case, the solution of Eqs. (15) and (16) are:

$$U_{cs}(t) = - e^{A_c t} \left(\int_{t}^{\infty} e^{-A_c s} B_{cs}(s)\, e^{A_s s}\, ds \right) e^{-A_s t} \tag{26}$$

$$U_{sc}(t) = e^{A_s t} \left(\int_{-\infty}^{t} e^{-A_s s} B_{sc}(s)\, e^{A_c s}\, ds \right) e^{-A_c t} \tag{27}$$

As before, the 2nd order correction to the drift from the stable mode is:

$$M_t\left\{E\left[\left(\int_{-\infty}^{0} e^{-A_c t}\, B_{cs}(t)\, e^{-A_s \tau}\, B_{sc}(t+\tau)\, e^{A_c(t+\tau)}\, d\tau\right) x_c\right]\right\} \tag{28}$$

which agrees with that obtained from the extended averaging theorem.

In both examples, we have assumed that the deterministic part has been reduced to a normal form. This assumption does not invalidate the equivalence in any way, since it has been shown that deterministic normal form and deterministic averaging methods are equivalent for non-nilpotent systems, Arnold [17] and Sethna [20].

In the above discussions, we have assumed that the linear operator is diagonalizable. However, when it is not diagonalizable, the method of averaging cannot be applied due to the fact that the matrix e^{At} contains terms that are polynomials in t and the time average does not exist. Unlike averaging, normal forms can be used in the analysis of such nilpotent systems. Results for a system with a double zero bifurcation is presented by Sri Namachchivaya [7] for a two dimensional case.

A comparison has been made between the stochastic normal forms and stochastic averaging, and the equivalence of these two methods is demonstrated for a special class of nonlinear stochastic systems [3]. It has been shown that for systems under the effect of linear multiplicative and additive noise, the stable modes lead to a second order correction in the critical modes which, to our knowledge, has been ignored by previous researchers. The results justify the viability of stochastic normal forms as an alternative to stochastic averaging, which to date has been a traditional technique in the analysis of weakly non-linear systems under broad-band excitation. Moreover, unlike stochastic averaging, the method of stochastic normal forms is not limited to systems with non-nilpotent linear parts.

3. Summary of Results of Completed Work

The work completed under this project is summarized in the following subsections. Detailed analyses of these problems are given in Appendices A-F. Most of the results of the initial investigation are presented in [6]. However, due to its great length, this paper has been omitted from the final report.

3.1 Stochastic Analysis of Nonconservative Systems

As a second step, using the above developed extended averaging method, the PI and his graduate student [2] investigated the dynamic stability of stochastically excited general linear nonconservative systems. Modified stochastic averaging method is employed to obtain the contribution from the stochastic components of the stable modes to that of the critical modes. Results of mean square stability are shown to depend only on those values of the excitation spectral density near twice, difference and combination of natural frequencies of the nonconservative system. The details are found in *Appendix A*. Subsequent research involved applying these techniques to obtain an understanding of co-dimension one [6,21] and co-dimension two [7] stochastic bifurcations that occur in nonlinear nonconservative systems.

Statistical properties of the stochastic response of a system undergoing a Hopf or simple bifurcation in the presence of parametric and external excitations have been obtained [6,21]. It was found that the addition of small stochastic parametric excitation gives rise to a shift in the bifurcation point whereas the addition of external excitation modifies the bifurcation behavior entirely. The result is shown in Fig. 1, where we observe that the *parametric excitation has a "stabilizing" effect*. That is, it delays the transition of the trivial solution (zero equilibrium state) from

being a stable solution to an unstable solution. Physically, this indicates that parametric excitation is not necessarily undesirable when considering system stability. Furthermore, it is shown that in multidegree-of-freedom systems the contribution of the stochastic components in the stable modes to the drift term of the critical mode may be beneficial in terms of stability.

The stationary probability density functions have been obtained. The effect of external excitation can also be seen by comparing Fig. 2 and Fig. 3. This distinguishing feature is reflected in the stationary moments of the system shown in Fig. 4. The above findings illustrate the non-intuitive behavior possible in stochastically perturbed nonlinear systems on the verge of bifurcation.

A dynamical system undergoes a co-dimension two bifurcation due to the presence of additional degeneracies other than those encountered for the simple and the Hopf bifurcations. In [7], attention is restricted to the stochastic version of the case of double zero eigenvalues with non-semi-simple forms. The case under investigation is that in which the normal form associated with non-semi-simple double zero eigenvalues is perturbed by weak Gaussian white noise. Moreover, since the normal form for this case represents the van der Pol - Duffing oscillator, it can be viewed as a van der Pol - Duffing oscillator under both parametric and external excitations. Detailed analysis of the stochastic normal forms of this co-dimension two bifurcation has been given by Sri Namachchivaya in *Appendix B*.

3.2 Applications

The major goals of this research effort were to examine the stochastic dynamics, stability and bifurcation behavior of various nonlinear stochastic problems with direct impact on the mission of AFOSR. Such problems

include: aircraft at high angles of attack, panels under gas flow with both turbulent boundary layers and random axial loads; rotating shafts and rotating systems under pulsating loads; propellant lines conveying turbulent flow, etc.

3.2.1 Aircraft at High Angles of Attack [22,23]

Consider an aircraft in steady flight at an angle of attack s. Suppose some disturbances take place at time t = 0, e.g., due to a change in the flap deflection angle; the aircraft will subsequently undergo an unsteady motion relative to its steady flight. Such an unsteady motion of the aircraft modifies the air flow and hence the aerodynamic forces on the aircraft which in turn determine its motion. Thus, the aircraft's subsequent motion can only be determined by simultaneously solving the unsteady flow equations of the air and the equations of motion of the vehicle as a rigid body, aeroelastic effects being assumed negligible.

Although simultaneously solving the coupled equations in principle represents an exact approach to the problem of arbitrary maneuvers, it is inevitably a very difficult and costly approach. In classical aerodynamics, the traditional approximate approach is to assume the pitching motion to be a small amplitude periodic oscillation consisting of simple harmonics. On this basis the flow equations are decoupled from the inertia equation, and are linearized to determine the aerodynamic response to such a harmonic motion. The so-called aerodynamic coefficients thus obtained are then used to predict the motion of the aircraft. Even though this approach ignores the time-history effects on the flow field and the aircraft motion, it gives a good approximation for calculating the aerodynamic response, and hence, the pitching moment from the unsteady flow equations. This approximation was adopted by the PI in his investigations [22,23] of this problem.

A complete unfolding of a co-dimension two bifurcation due to a double zero eigenvalue of the equations of pitching motion of an aircraft was carried out in the vicinity of zero stiffness derivative, and zero damping derivative. Unfolding of such a singularity uncovered all possible bifurcations that were present in the vicinity of the singularity, in addition to the results of various other previous investigations. This method provided results pertaining to uniqueness of limit cycles and global bifurcations. A detailed analysis of this problem is presented in *Appendix C*.

The analysis of post-critical behavior of aircraft based solely on deterministic nonlinear analysis has not proven to be adequate. The inclusion of the effects of a turbulent atmosphere increases the sophistication of the model. It is possible to regard isotropic atmospheric turbulence as a broadband stochastic process. The nonlinear system now has stochastic elements. *Appendix D* contains a broader description and demonstrates the methodology by investigating the effects of atmospheric turbulence on the lateral dynamics of fighter aircraft at large angles of attack and sideslip.

3.2.2 Rotating Shaft [1]

One of the most fundamental components of a mechanical system is a rotating shaft. It is, therefore, not surprising that through the years considerable effort has been directed at obtaining a better understanding of such mechanisms. Toward this end, an analytical method, based on some of the mathematical ideas mentioned above, has been applied for investigating a rotating shaft under stochastic excitations of small intensity. Explicit stability conditions are derived for first and second moments of a two degree-of-freedom rotating shaft. When the stochastic

excitation is a white noise excitation, the first moment stability conditions reduce to that of the deterministic case. It is shown that the addition of non-white noise excitation has a stabilizing effect on the parametric instability of harmonically excited rotating shafts. Finally, the stability conditions of a symmetric shaft along with their numerical results are presented. The mean square stability conditions for purely white noise excitation are given in Fig. 5(a) and 5(b), where S_0, D and W are the spectral density, nondimensional damping and shaft rotational speed, respectively. The $\bar{\omega}$ represents the normalized natural frequency of the symmetric shaft. Explicit results for both the white and non-white noise cases are detailed in *Appendix E*.

3.2.3 Propellant Lines Conveying Pulsating Fluid [24,25]

The transverse vibration of propellant lines of large liquid-fuel rockets and other vehicles continues to be a problem for the space industry. Here it is realistic to assume that the flow is turbulent and the support excitations are random. The deterministic problem with harmonic flow velocity has been investigated as a preliminary step [24]. In this preliminary study, the ideas related to the method of averaging, Poincare-Birkoff normal forms, and the center manifold theorem have been used at different stages of the analysis to investigate the stability and bifurcation behavior of nonlinear supported pipes conveying pulsating fluid. Explicit results for the stability boundaries of the trivial solution, as well as bifurcating paths and their stability, have been obtained for values of the system parameters m (amplitude of the excitation) and u (frequency of the excitation), where the value of u is taken in the neighborhood of subharmonic and combination resonances. These results are presented in *Appendix F*. In the subsequent study, the PI included the effect of random

excitation and obtained results for mean square stability. The detailed analysis is given in [25].

4. List of Refereed Publications from this Project

1. N. Sri Namachchivaya, "Stochastic bifurcation," Journal of Applied Mathematics and Computations, Vol. 38, 1990, pp. 101-159.
2. N. Sri Namachchivaya and G.S.B. Leng, "Equivalence of stochastic averaging and stochastic normal forms," Journal of Applied Mechanics (ASME) (in press).
3. N. Sri Namachchivaya and W. M. Tien, "Stochastically excited linear nonconservative systems," Journal of Mechanics of Structures and Machines (in press).
4. H. H. Hilton and N. Sri Namachchivaya, "Effects of noise in nonlinear systems exhibiting simple bifurcation," Journal of Structural Safety, Vol. 6(2), 1989, pp. 211-221.
5. N. Sri Namachchivaya, "Co-dimension two bifurcation in the presence of noise," Journal of Applied Mechanics (ASME) (in press).
6. N. Sri Namachchivaya and H. J. Van Roessel, "Unfolding of bifurcations associated with double zero eigenvalue for supersonic flow past a pitching wedge," AIAA Journal of Guidance, Control and Dynamics, 1990, pp. 343-347.
7. N. Sri Namachchivaya, "Mean-square stability of a rotating shaft under combined harmonic and stochastic excitations," Journal of Sound and Vibration, Vol. 133(2), 1989, pp. 323-336.
8. N. Sri Namachchivaya and W. M. Tien, "Bifurcation behavior of nonlinear pipes conveying pulsating flow," Journal of Fluid and Structures, Vol. 3(4), 1989, pp. 81-102.

9. N. Sri Namachchivaya and H. H. Hilton, "Stochastic stability of supported pipes conveying pulsating fluid," Current Topics in Structural Mechanics (Howard Chung, ed., ASME, Vol. 179) pp. 75-80, 1989.

5. Supervised Graduate Students and Other Personnel Supported by the AFOSR Award

1) Student Name: Gerard S. Leng (Aug. 1988 - Aug. 1990)

   Degree: Ph.D.

   Thesis Title: Mode Interaction of Dynamical Systems in the Presence of External Random Excitations

   Date of Completion: August 1990

2) Student Name: Win Min Tien (Aug. 1988 - Aug. 1990)

   Degree: Ph.D.

   Thesis Title: Co-dimension Two Stochastic Bifurcation with Application to Panel Flutter

   Date of Completion: August 1992

3) Student Name: Sanjiv Talwar (July 1990 - Aug. 1990)

   Degree: Ph.D.

   Thesis Title: Nonlinear Control of Distributed Parameter Systems

   Date of Completion: December 1992

4) Student Name: Monica Doyle (Jan. 1990 - Aug. 1990)

   Degree: M.S.

   Thesis Title: Stability and Bifurcation Behavior of Nonlinear Rotating Systems

   Date of Completion: December 1990

5) Visiting Assistant Professor: H. J. Van Roessel

   Duration: August 21 - September 21, 1988

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APPENDIX A

I. Introduction

This paper investigates the stability of the trivial solution of linear nonconservative structural/mechanical systems with stochastically varying parameters. The stability of a single degree of freedom linear stochastic system has been studied by several investigators. Notably, Stratonovich and Romanovskii [1], Weidenhammer [2], Graefe [3] found that stability depends only on the excitation spectral density at frequencies near twice the system's natural frequency, a result analogous to that for the deterministic Mathieu equation. These results have been extended by Ariaratnam and Srikantaiah [4] and Sri Namachchivaya and Ariaratnam [5] for general multidegree of freedom conservative nongyroscopic and gyroscopic systems, respectively. In this paper, we deal with oscillatory multidegree of freedom linear systems with gyroscopic and circulatory forces and it is assumed that a finite number of modes undergo flutter instability, while the remaining modes are assumed stable. For systems examined in [4,5] the method of stochastic averaging, which was initially proposed by Stratonovich [6], has been used to obtain conditions for stability in the second norm of the response. However, for the system under consideration, both the rapidly oscillating flutter modes and decaying stable modes are coupled with rapidly varying stochastic components. The asymptotic behavior of such a system is studied using the modified method of averaging [7,8].

II. Equation of Motion

The equations of motion of a linear nonconservative system subjected to random parametric and external excitation of small intensity can be written in the matrix form

$$A\,\ddot{q} + B\,\dot{q} + C\,q = \varepsilon F(q,\dot{q},t) \tag{1}$$

In equation (1), the n vector q represents the generalized coordinates of the

system, A, B and C are constant nxn matrices, F represents an n vector linear term in g and $\dot{g}$, i.e.

$$\underset{\sim}{F}(g,\dot{g},t) = (D_1 g + D_2 \dot{g}) f(t) + \underset{\sim}{s} g(t)$$

Furthermore, the matrix $A = A^T$ represents mass-like terms and is generally positive definite. The matrix $B = \mu(D+G)$, where D represents linear energy dissipation terms and the matrix G represents the gyroscopic terms arising usually from the Coriolis forces, i.e. $G = -G^T$. The matrix $C = K + \mu(K_1 + K_2)$, where $K = K^T$ and $K_1 = K_1{}^T$ are stems from the potential energy and the centrifugal forces respectively, and K_2 is an antisymmetric matrix. The parameter μ is usually referred to as loading parameter in mechanical system. The term f(t) and g(t) represents the time dependent stochastic perturbations. Moreover, the matrices D_1, D_2 are some nxn constant matrices. It may be noted that the unperturbed linear (ε=0) equations, when both D and K_2 are identically zero, represent a conservative gyroscopic system which has been studied by Sri Namachchivaya and Ariaratnam [5].

It is convenient to transform equation (1) into a vector form of 2n-dimension. Putting

$$R = \begin{bmatrix} 0 & A \\ A & B \end{bmatrix}_{2nx2n} \qquad S = \begin{bmatrix} A & 0 \\ 0 & -C \end{bmatrix}_{2nx2n} \qquad Q = \begin{bmatrix} 0 & 0 \\ D_2 & D_1 \end{bmatrix}_{2nx2n}$$

$$\underset{\sim}{s} = \begin{bmatrix} 0 \\ \underset{\sim}{s}_o \end{bmatrix}_{2nx1} \quad \text{and} \quad \underset{\sim}{x} = \begin{bmatrix} \dot{g} \\ g \end{bmatrix}_{2nx1}$$

now equations (1) can be transformed into

$$R\dot{\underset{\sim}{x}} = S\underset{\sim}{x} + \varepsilon\, Q\underset{\sim}{x}\, f(t) + \varepsilon\, \underset{\sim}{s}\, g(t) \qquad (2)$$

It may be noted that R and S are chosen so that they are symmetric when A, B

and C are symmetric. In general, the unperturbed system $R\dot{\underset{\sim}{x}} = S\underset{\sim}{x}$, has 2n eigenvalues, some of which are complex conjugate pairs $\lambda_j = \delta_j + i\omega_j$, $\lambda_{N+j} = \delta_j - i\omega_j$, $j = 1,2 \,..\, N$ and the rest may be real eigenvalues $\lambda_{2N+j} = \alpha_j$, $j = 1,2..m$, such that $2n = 2N + m$. It is obvious that when all the real parts of the eigenvalues are negative and large, the unperturbed system will be asymptotically stable and the small stochastic perturbations, i.e. f(t) and g(t) will have negligible effect on the system. However, as the loading parameter, μ, varies and reaches a neighborhood of certain critical value, some of the real parts of the eigenvalues approach zero and the modes corresponding to these eigenvalues are sensitive to small perturbations. In this paper, the effect of stochastic fluctuations on linear nonconservative system (2) undergoing a "flutter" type instability is studied.

Consider the system (2), it is assumed that at $\mu = \mu_{cr}$ the system contains M pairs of noncoincident pure imaginary eigenvalues with no resonance i.e. $n_1\omega_1 + n_2\omega_2 + \,..\, n_M\omega_M \neq 0$ ($n_j \gtrless 0$, integers), and the remaining eigenvalues have negative real parts. To study the behavior for small derivatives from μ_{cr}, let $\mu = \mu_{cr} + \eta$, and the unperturbed part of equations (2) is brought to the simplest form that has a minimum coupling to this end, consider the transformation [9]

$$\underset{\sim}{x} = \bar{C}\underset{\sim}{y}^T \quad , \quad \underset{\sim}{y} = [\underset{\sim}{U}\ \underset{\sim}{V}\ \underset{\sim}{W}] \tag{3}$$

$$\bar{C} = [2c^1,\ 2d^1,\ 2c^2,\ 2d^2\ \ldots\ a^{2N+1} \ldots\ldots a^{2n}]$$

where $\underset{\sim}{U}$, $\underset{\sim}{V}$ and $\underset{\sim}{W}$ are 2M 2(N-M) and $m = 2(n-N)$ dimensional vectors, respectively and $\bar{C}$ consists of column vectors that are the real and imaginary parts of the eigenvectors $\underset{\sim}{a} = \underset{\sim}{c} + i\underset{\sim}{d}$ of the direct eigenvalues problem. Furthermore, for convenience, the eigenvectors have been ordered such that the first M eigenvectors correspond to the critical eigenvalues. Substituting (3)

into (2), and premultiplying by the adjoint eigenvectors

$$D = [\underset{\sim}{e}^1, -\underset{\sim}{f}^1, \underset{\sim}{e}^2, -\underset{\sim}{f}^2, \; .. \; b^{2N+1}, .. \; b^{2n}]$$

where $\underset{\sim}{b}$ is the adjoint eigenvector given by $\underset{\sim}{b} = \underset{\sim}{e} + i\underset{\sim}{f}$ and $\underset{\sim}{a}^i \underset{\sim}{b}^j = \delta_{ij}$ yields

$$\begin{aligned}
\dot{\underset{\sim}{U}} &= A_o \underset{\sim}{U} + \varepsilon[\underset{\sim}{K}^o U + \underset{\sim}{M}^1 V + \underset{\sim}{M}^2 W] f(t) + \varepsilon s^{(1)} g(t) \\
\dot{\underset{\sim}{V}} &= B_o \underset{\sim}{V} + \varepsilon[N^1 \underset{\sim}{U} + L^1 \underset{\sim}{V} + L^2 \underset{\sim}{W}] f(t) + \varepsilon s^{(2)} g(t) \\
\dot{\underset{\sim}{W}} &= C_o \underset{\sim}{W} + \varepsilon[N^2 \underset{\sim}{U} + P^1 \underset{\sim}{V} + P^2 \underset{\sim}{W}] f(t) + \varepsilon s^{(3)} g(t)
\end{aligned} \tag{4}$$

where $A_o = \text{diag} \begin{bmatrix} \Delta\delta_j & \omega_j \\ -\omega_j & \Delta\delta_j \end{bmatrix}, \quad j = 1,2 \; .. \; M$

$$B_o = \text{diag} \begin{bmatrix} -\delta_j & \omega_j \\ -\omega_j & -\delta_j \end{bmatrix}, \quad j = M+1, \; M+2 \; .., \; N$$

$$C_o = \text{diag} \left[-\alpha_1, \; -\alpha_2, \; \ldots \; -\alpha_m \right], \quad m = 2(n-N)$$

$$\begin{bmatrix} K^o & M^1 & M^2 \\ N^1 & L^1 & L^2 \\ N^2 & P^1 & P^2 \end{bmatrix} = D^T Q \bar{C}, \quad \Delta\delta_j \sim O(n)$$

and the matrices K, L, M, N and P are explicitly given in Appendix A. Using the transformation, $\underset{\sim}{V} = T \underset{\sim}{Z}$

where $T = \left[\text{diag} \begin{bmatrix} \cos(\omega_j t) & \sin(\omega_j t) \\ -\sin(\omega_j t) & \cos(\omega_j t) \end{bmatrix} \right], \quad j = M+1, \quad M+2, \ldots N$

the system (4) is transformed into the form as

$$\dot{U}_{2j-1} - \omega_j U_{2j} = \epsilon\{\sum_{k=1}^{M}[K^{o}_{2j-1,2k-1}U_{2k-1} + K^{o}_{2j-1,2k}U_{2k}]$$

$$+ \sum_{k=1}^{N-M}[(M^{1}_{2j-1,2k-1}\cos(\omega_{M+k}t) - M^{1}_{2j-1,2k}\sin(\omega_{M+k}t))Z_{2k-1}$$

$$+ (M^{1}_{2j-1,2k-1}\sin(\omega_{M+k}t) + M^{1}_{2j-1,2k}\cos(\omega_{M+k}t))Z_{2k}]$$

$$+ \sum_{k=1}^{m}[M^{2}_{2j-1,k}W_k]\}f(t) + \epsilon s^{(1)}_{2j-1}g(t) + \epsilon^2\Delta\ \delta_j U_{2j-1}$$

$$\dot{U}_{2j} + \omega_j U_{2j-1} = \epsilon\{\sum_{k=1}^{M}[K^{o}_{2j-1,2k-1}U_{2k-1} + K^{o}_{2j,2k}U_{2k}] \tag{5}$$

$$+ \sum_{k=1}^{N-M}[(M^{1}_{2j,2k-1}\cos(\omega_{M+k}t) - M^{1}_{2j,2k}\sin(\omega_{M+1}t))Z_{2k-1}$$

$$+ (M^{1}_{2j,2k-1}\sin(\omega_{M+k}t) + M^{1}_{2j,2k}\cos(\omega_{M+k}t))Z_{2k}]$$

$$+ \sum_{k=1}^{m}[M^{2}_{2j,k}W_k]\}f(t) + \epsilon s^{(1)}_{2j}g(t) + \epsilon^2\Delta\ \delta_j U_{2j}\ , \quad j=1,2\ldots M$$

$$\dot{Z}_{2r-1} = -\delta_r Z_{2r-1} + \epsilon\{\sum_{s=1}^{M}[(N^{1}_{2r-1,2s-1}\cos(\omega_{M+r}t) - N^{1}_{2r,2s-1}\sin(\omega_{M+r}t))U_{2s-1}$$

$$+ (N^{1}_{2r-1,2s}\cos(\omega_{M+r}t) - N^{1}_{2r,2s}\sin(\omega_{M+r}t))U_{2s}]$$

$$+ \psi^{(1)}_{2r-1}(\underline{Z},t) + \psi^{(2)}_{2r-1}(\underline{W},t)\}\ f(t)$$

$$+ \epsilon[s^{(2)}_{2r-1}\cos(\omega_{M+r}t) - s^{(2)}_{2r}\sin(\omega_{M+r}t)]\ g(t)$$

$$\dot{Z}_{2r} = -\ \delta_r Z_{2r} + \epsilon\{\sum_{s=1}^{M}[(N^{1}_{2r-1,2s-1}\sin(\omega_{M+r}t) + N^{1}_{2r,2s-1}\cos(\omega_{M+r}t))U_{2s-1}$$

$$+ \left(N^{1}_{2r-1,2s}\sin(\omega_{M+r}t) + N^{1}_{2r,2s}\cos(\omega_{M+r}t)\right)U_{2s}\Big]$$

$$+ \psi^{(1)}_{2r}(\underset{\sim}{Z},t) + \psi^{(2)}_{2r}(\underset{\sim}{W},t)\Big\}\ f(t)$$

$$+ \varepsilon\left[s^{(2)}_{2r-1}\sin(\omega_{M+r}t) + s^{(2)}_{2r}\cos(\omega_{M+r}t)\right]\ g(t)\ , \quad r = 1,2,\ldots N-M$$

where

$$\psi^{(1)} = T^{-1}L^{1}T\,\underset{\sim}{Z} \quad , \quad \psi^{(2)} = T^{-1}L^{2}\underset{\sim}{W}\ .$$

$$\dot{W}_i = -\alpha_i W_i + \varepsilon\Big\{\sum_{\ell=1}^{M}\left[N^{2}_{i,2\ell-1}U_{2\ell-1} + N^{2}_{i,2\ell}U_{2\ell}\right]$$

$$+ \bar{\psi}^{(1)}(\underset{\sim}{Z},t) + \bar{\psi}^{(2)}(\underset{\sim}{W},t)\Big\}\ f(t)$$

$$+ \varepsilon\ s^{(3)}_{i}\ g(t)\ , \quad i = 1,2\ldots m$$

where $\bar{\psi}^{(1)} = P^{1}T\underset{\sim}{Z}$, $\bar{\psi}^{(2)} = P^{2}\underset{\sim}{W}$.

the first 2M equations are critical and the rest n-2M equations are asymptotically stable, i.e. $\delta_r \gg 0$, $\alpha_i \gg 0$, $r = 1,2,\ldots,$ M-N and $i = 1,2\ldots m$.

In physical systems, the excitations are real noise processes and results under the assumptions of white noise excitations are not directly applicable. However, for excitation processes with wide band spectrum of nearly constant spectral density, the equation of motion may be approximated by a set of equivalent Itô equations whose solution is a Markov process and satisfies the Fokker-Planck equation. The results obtained may then be applied to certain real physical systems. This approximation is made by applying the method of stochastic averaging to a set of equations in "standard form" that are exactly

equivalent to the equations of motion (5). Such equations in standard form are achieved by means of the transformation

$$U_{2j-1} = a_j \sin\Phi_j \quad , \quad U_{2j} = a_j \cos\Phi_j \qquad (6)$$

where

$$\Phi_j = \omega_j t + \phi_j \quad , \quad j = 1,2,\ldots M$$

Substitution of equation (6) in equation (5) yields a set of 2n first order equations in $\underline{a}$, $\underline{\phi}$, $\underline{Z}$ and $\underline{W}$ in the form

$$\dot{a}_j = \varepsilon[\varepsilon F_j^0(\underline{a}) + F_j^1(\underline{a},\underline{\phi},\underline{Z},\underline{W},t)f(t) + G_j^1(\underline{\phi},t)g(t)]$$

$$\dot{\phi}_j = \varepsilon[F_j^2(\underline{a},\underline{\phi},\underline{Z},\underline{W},t)f(t) + G_j^2(\underline{\phi},t)g(t)]/a_j \ , \quad j = 1,2\ldots M$$

$$\dot{Z}_{2r-1} = -\delta_r Z_{2r-1} + \varepsilon[F_{2r-1}^3(\underline{a},\underline{\phi},\underline{Z},\underline{W},t)f(t) + G_{2r-1}^3(t)g(t)] \qquad (7)$$

$$\dot{Z}_{2r} = -\delta_r Z_{2r} + \varepsilon[F_{2r}^3(\underline{a},\underline{\phi},\underline{Z},\underline{W},t)f(t) + G_{2r}^3(t)g(t)] \ , \quad r = 1,2,\ldots N-M$$

$$\dot{W}_k = -\alpha_k W_k + \varepsilon[F_k^4(\underline{a},\underline{\phi},\underline{Z},\underline{W},t)f(t) + G_k^4(t)g(t)] \ , \quad k = 1,2\ldots m$$

where

$$F_j^0 = \Delta\delta_j a_j \ ,$$

$$F_j^1 = X_{a_j}^{(0)}(a,\phi,t) + X_{a_j}^{(1)}(Z,\phi,t) + X_{a_j}^{(2)}(W,\phi,t) \ ,$$

$$G_j^1 = s_{2j-1}^{(1)}\sin\Phi_j + s_{2j}^{(1)}\cos\Phi_j \ ,$$

$$F_j^2 = X_{\phi_j}^{(0)}(a,\phi,t) + X_{\phi_j}^{(1)}(Z,\phi,t) + X_{\phi_j}^{(2)}(W,\phi,t) \ ,$$

$$G_j^2 = s_{2j-1}^{(1)}\cos\Phi_j - s_{2j}^{(1)}\sin\Phi_j \ ,$$

$$F_{2r-1}^3 = X_{Z_{2r-1}}^{(0)}(a,\phi,t) + \psi_{2r-1}^{(1)}(Z,t) + \psi_{2r}^{(2)}(W,t) \ ,$$

$$G_{2r-1}^3 = s_{2r-1}^{(2)}\cos(\omega_{M+r}t) - s_{2r}^{(2)}\sin(\omega_{M+r}t) \ ,$$

$$F^3_{2r} = X^{(0)}_{Z_{2r}}(a,\phi,t) + \psi^{(1)}_{2r}(Z,t) + \psi^{(2)}_{2r}(W,t) \, ,$$

$$G^3_{2r} = s^{(2)}_{2r-1}\sin(\omega_{M+r}t) + s^{(2)}_{2r}\cos(\omega_{M+r}t) \, ,$$

$$F^4_{k_i} = X^{(0)}_{W_i}(a,\phi,t) + \bar{\psi}^{(1)}_i(Z,t) + \bar{\psi}^{(2)}_i(W,t) \, ,$$

$$G^4_{k_i} = s^{(3)}_{k_i} \, ,$$

the matrices $X^{(i)}_a$, $X^{(i)}_\phi$, $X^{(i)}_W$ are explicitly given in Appendix B.

III. Approximation to Markov Process

An exact solution of Eqs. (7) for arbitrary random processes $f(t)$ and $g(t)$ is not available. However, if the intensities as well as the correlation time of the processes $f(t)$ and $g(t)$ are sufficiently small, using a limit theorem due to Papanicolaou and Kohler [7,10], the solution process $\{a(t), \phi(t)\}$ may be approximated in the weak sense by a diffusive Markov vector process. It is then possible to solve for the response amplitudes. This method of approximation, known as stochastic averaging is analogous to the ordinary averaging method of Bogoliubov and Mitropolsky [11] for the deterministic case.

By making use of the formulas of Appendix C as shown in Papanicolaou and Kohler [7,10], one can obtain the drift and diffusion terms, however, evaluation of the drift and diffusion coefficient defined in Appendix C is long and cumbersome. Bypassing some of the algebraic details, drift and diffusion terms are given as

$$m_{a_r} =$$

$$\Delta\delta_r a_r + \frac{a_r}{8} J^{+2}_{1,rr} S_{ff}(0) + \frac{3a_r}{16}\left(H^{+2}_{1,rr} + J^{-2}_{1,rr}\right)S_{ff}(2\omega_r)$$

$$+ \frac{1}{4a_r}\left({s^1_{2r-1}}^2 + {s^1_{2r}}^2\right)S_{gg}(\omega_r)$$

$$+ \sum_{s \neq r}^{M} \frac{a_s^2}{16a_r} [(H_{1,rs}^{+2} + J_{1,rs}^{-2}) S_{ff}(\omega_r + \omega_s)$$

$$+ (H_{1,rs}^{-2} + J_{1,rs}^{+2}) S_{ff}(\omega_r - \omega_s)]$$

$$+ \sum_{s \neq r}^{M} \frac{a_r}{8} [(H_{1,rs}^{+} H_{1,sr}^{+} + J_{1,rs}^{-} J_{1,sr}^{-}) S_{ff}(\omega_r + \omega_s)$$

$$- (H_{1,rs}^{-} H_{1,sr}^{-} - J_{1,rs}^{+} J_{1,sr}^{+}) S_{ff}(\omega_r - \omega_s)$$

$$+ (H_{1,sr}^{-} J_{1,rs}^{+} + H_{1,rs}^{-} J_{1,sr}^{+}) \psi_{ff}(\omega_r - \omega_s)$$

$$- (H_{1,sr}^{+} J_{1,rs}^{-} - H_{1,rs}^{+} J_{1,sr}^{-}) \psi_{ff}(\omega_r + \omega_s)]$$

$$+ \sum_{s=1}^{N-M} \frac{a_r}{8} [(H_{2,rs}^{+} H_{3,sr}^{+} + J_{2,rs}^{-} J_{3,sr}^{-}) \bar{S}_{ff}(\omega_r + \omega_{M+s})$$

$$- (H_{2,rs}^{-} H_{3,rs}^{-} - J_{2,rs}^{+} J_{3,sr}^{+}) \bar{S}_{ff}(\omega_r - \omega_{M+s})$$

$$+ (H_{3,sr}^{-} J_{2,rs}^{+} + H_{2,rs}^{-} J_{3,sr}^{+}) \bar{\psi}_{ff}(\omega_r - \omega_{M+s})$$

$$- (H_{3,sr}^{+} J_{2,rs}^{-} - H_{2,rs}^{+} J_{3,sr}^{-}) \bar{\psi}_{ff}(\omega_r + \omega_{M+s})]$$

$$+ \sum_{s=1}^{m} \frac{a_r}{8} [(M_{2r-1,s}^{2} N_{s,2r-1}^{2} + M_{2r,s}^{2} N_{s,2r}^{2}) \hat{S}_{f}(\omega_r)$$

$$- (M_{2r,s}^{2} N_{s,2r-1}^{2} - M_{2r-1,s}^{2} N_{s,2r}^{2}) \hat{\psi}_{ff}(\omega_r)]$$

$^{m}\phi_r$

$$= -\frac{1}{8}\left(H^{+\,2}_{1,rr} + J^{-\,2}_{1,rr}\right)\psi_{ff}(2\omega_r)$$

$$+ \frac{1}{8}\sum_{s\neq r}^{M}\left[-\left(H^{+}_{1,sr}J^{-}_{1,rs} - H^{+}_{1,rs}J^{-}_{1,sr}\right)S_{ff}(\omega_r + \omega_s)\right.$$

$$+ \left(H^{-}_{1,sr}J^{+}_{1,rs} + H^{-}_{1,rs}J^{+}_{1,sr}\right)S_{ff}(\omega_r - \omega_s)$$

$$- \left(H^{+}_{1,rs}H^{+}_{1,sr} + J^{-}_{1,rs}J^{-}_{1,sr}\right)\psi_{ff}(\omega_r + \omega_s)$$

$$\left. + \left(H^{-}_{1,rs}H^{-}_{1,sr} - J^{+}_{1,rs}J^{+}_{1,sr}\right)\psi_{ff}(\omega_r - \omega_s)\right]$$

$$+ \frac{1}{8}\sum_{s=1}^{N-M}\left[-\left(H^{+}_{2,rs}H^{+}_{3,sr} + J^{-}_{2,rs}J^{-}_{3,sr}\right)\bar{\psi}_{ff}(\omega_r + \omega_{M+s})\right.$$

$$+ \left(H^{-}_{2,rs}H^{-}_{3,sr} - J^{+}_{2,rs}J^{+}_{3,sr}\right)\bar{\psi}_{ff}(\omega_r - \omega_{M+s})$$

$$+ \left(H^{-}_{3,sr}J^{+}_{2,rs} + H^{-}_{2,rs}J^{+}_{3,sr}\right)\bar{S}_{ff}(\omega_r - \omega_{M+s})$$

$$\left. - \left(H^{+}_{3,rs}J^{-}_{2,sr} - H^{+}_{2,rs}J^{-}_{3,sr}\right)\bar{S}_{ff}(\omega_r + \omega_{M+s})\right]$$

$$- \frac{1}{8}\sum_{s}^{m}\left[\left(M^{2}_{2r-1,s}N^{2}_{s,2r-1} + M^{2}_{2r,s}N^{2}_{s,2r}\right)\hat{\psi}_{ff}(\omega_r)\right.$$

$$\left. + \left(M^{2}_{2r,s}N^{2}_{s,2r-1} - M^{2}_{2r-1,s}N^{2}_{s,2r}\right)\hat{S}_{ff}(\omega_r)\right]$$

$$[\sigma\sigma^T]_{a_r a_r} =$$

$$\frac{s^{1\,2}_{2r-1} + s^{1\,2}_{2r}}{2}\, S_{gg}(\omega_r) + \frac{a_r^{\,2}}{4}\, J^{+\,2}_{1,rr}\, S_{ff}(0)$$

$$+ \frac{a_r^{\,2}}{8}\left(H^{+\,2}_{1,rr} + J^{-\,2}_{1,rr}\right) S_{ff}(2\omega_r)$$

$$+ \sum_{s \neq r}^{M} \frac{a_s^2}{8} [(H^{+2}_{1,rs} + J^{-2}_{1,rs}) S_{ff}(\omega_r + \omega_s)$$

$$+ (H^{-2}_{1,rs} + J^{+2}_{1,rs}) \ S_{ff}(\omega_r - \omega_s)]$$

$$[\sigma\sigma^T]_{a_r a_s} =$$

$$\frac{a_r a_s}{4} J^{+}_{1,rr} J^{+}_{ss} \ S_{ff}(0)$$

$$+ \ \frac{a_r a_s}{8} (H^{+}_{1,rs} H^{+}_{1,sr} + J^{-}_{1,rs} J^{-}_{1,sr}) \ S_{ff}(\omega_r + \omega_s)$$

$$- \frac{a_r a_s}{8} (H^{-}_{1,rs} H^{-}_{1,sr} - J^{+}_{1,rs} J^{+}_{1,sr}) \ S_{ff}(\omega_r - \omega_s)$$

$$[\sigma\sigma^T]_{a_r \phi_r} =$$

$$\frac{a_r}{4} H^{-}_{1,rr} J^{+}_{1,rr} \ S_{ff}(0)$$

$$[\sigma\sigma^T]_{a_r \phi_s} =$$

$$\frac{a_r}{4} H^{-}_{1,ss} J^{+}_{1,rr} \ S_{ff}(0)$$

$$+ \frac{a_r}{8} (H^{-}_{1,sr} J^{+}_{1,rs} + H^{-}_{1,rs} J^{+}_{1,sr}) \ S_{ff}(\omega_r - \omega_s)$$

$$+ \frac{a_r}{8} (H^{+}_{1,sr} J^{-}_{1,rs} - H^{+}_{1,rs} J^{-}_{1,sr}) \ S_{ff}(\omega_r + \omega_s)$$

$$[\sigma\sigma^T]_{\phi_r \phi_r} =$$

$$\frac{1}{4} H^{-2}_{1,rr} \ S_{ff}(0) + \frac{s^{1\,2}_{2r-1} + s^{1\,2}_{2r}}{2a_r^{\,2}} \ S_{gg}(\omega_r)$$

$$+ \frac{1}{8} \left(H^{+2}_{1,rr} + H^{-2}_{1,rr}\right) S_{ff}(2\omega_r)$$

$$+ \sum_{s \neq r}^{M} \frac{a_s^2}{8a_r^2} \left[\left(H^{+2}_{1,rs} + J^{-2}_{1,rs}\right) S_{ff}(\omega_r + \omega_s)\right.$$

$$\left. + \left(H^{-2}_{1,rs} + J^{+2}_{1,rs}\right) S_{ff}(\omega_r - \omega_s)\right]$$

$$[\sigma\sigma^T]_{\phi_r\phi_s} =$$

$$\frac{1}{4} H^-_{1,rr} H^-_{1,ss} S_{ff}(0)$$

$$+ \frac{1}{8} \left(H^+_{1,rs} H^+_{1,sr} + J^-_{1,rs} J^-_{1,sr}\right) S_{ff}(\omega_r + \omega_s)$$

$$+ \frac{1}{8} \left(H^-_{1,rs} H^-_{1,sr} - J^+_{1,rs} J^+_{1,sr}\right) S_{ff}(\omega_r - \omega_s)$$

where

$$H^{\pm}_{1,jk} = K^{o}_{2j-1,2k} \pm K^{o}_{2j,2k-1} , \quad J^{\pm}_{1,jk} = K^{o}_{2j,2k} \pm K^{o}_{2j-1,2k-1} ,$$

$$H^{\pm}_{2,jk} = M^{1}_{2j-1,2k} \pm M^{1}_{2j,2k-1} , \quad J^{\pm}_{2,jk} = M^{1}_{2j,2k} \pm M^{1}_{2j-1,2k-1} ,$$

$$H^{\pm}_{3,rs} = N^{1}_{2r-1,2s} \pm N^{1}_{2r,2s-1} , \quad J^{\pm}_{3,rs} = N^{1}_{2r,2s} \pm N^{1}_{2r-1,2s-1} .$$

$$S_{ff}(\omega) = 2\int_0^\infty R_{ff}(\tau)\cos\omega\tau d\tau , \quad \Psi_{ff}(\omega) = 2\int_0^\infty R_{ff}(\tau)\sin\omega\tau d\tau ,$$

$$\bar{S}_{ff}(\omega) = 2 \int_0^\infty R_{ff}(\tau)\, e^{-\delta_i\tau} \cos\omega\tau d\tau ,$$

$$\bar{\Psi}_{ff}(\omega) = 2\int_0^\infty R_{ff}(\tau)\, e^{-\delta_i\tau} \cos\omega\tau d\tau , \quad i = 1,2,\ldots,N-M$$

$$\hat{S}_{ff}(\omega) = 2\int_0^\infty R_{ff}(\tau)\, e^{-\alpha_j\tau} \cos\omega\tau d\tau ,$$

$$\hat{\Psi}_{ff}(\omega) = 2\int_0^\infty R_{ff}(\tau)\, e^{-\alpha_j \tau} \sin\omega\tau d\tau \ . \quad j = 1,2,\ldots,m$$

IV. Stability Analysis

When the excitation is deterministic, the concept of stability is the usual Liapunov concept employed in the theory of stability of motion, while in the case of stochastic excitation, a definition of stability must be introduced. There are several definitions of stochastic stability in the literature but most investigations have been concerned with only two types; namely, stability in the moments and almost sure or sample stability. In this paper, we shall examine only the moment stability of the trivial solution.

The trivial solution $\underset{\sim}{a} = 0$ of the system

$$\dot{\underset{\sim}{a}} = F\big(\underset{\sim}{a}, t, \underset{\sim}{\eta}(t)\big) = 0 \ , \quad F\big(o, t, \underset{\sim}{\eta}(t)\big) = 0 \tag{8}$$

is said to be asymptotically stable in the nth moment if all joint moment of order n of the component of $\underset{\sim}{a}$ are bounded in absolute value for $t \geq 0$. For n=1, the trivial solution is said to be stable in the mean. For n=2, the trivial solution is said to be stable in mean square.

A suitable norm $||y(t)||$ of the state vector in the critical mode $\underset{\sim}{u}$ is defined and conditions are derived such that $E[||y(t)||^2]$ remains bounded for t tending to infinity. The norm of the solution vector $\underset{\sim}{U}$ is defined by

$$||y|| = \Big[\sum_{r=1}^{M} \left(U_{2r-1}^2 + U_{2r}^2\right)\Big]^{1/2} = \Big[\sum_{r=1}^{M} a_r^2\Big]^{1/2}$$

so that

$$E[||y||^2] = \sum_{r=1}^{M} E[a_r^2] = \sum_{r=1}^{M} \bar{M}_r$$

where $\bar{M}_r(t)$ denoted the second moment of the amplitude a_r. Hence,

$$E[||y||^2] \to 0 \quad \text{if} \quad \bar{M}_r \to 0 \,, \qquad r = 1,2,\ldots M$$

By using the Itô differential rule [12]; namely, if $\xi(\underline{a},t)$ denotes any twice differentiable scalar function of $\underline{a}$, t then the corresponding Itô equation for $\xi(\underline{a},t)$ is

$$d\xi = L(\xi)dt + \varepsilon \sum_{i=1}^{M} \sum_{j=1}^{M} \sigma_{ij} \frac{\partial \xi}{\partial a_i} dw_j \tag{9}$$

where $L(\cdot)$ is the differential operator

$$L(\cdot) = \frac{\partial}{\partial t}(\cdot) + \varepsilon^2 \sum_{i=1}^{n} M_{a_i} \frac{\partial}{\partial a_i}(\cdot) + \varepsilon^2 \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\sigma\sigma^T]_{ij} \frac{\partial^2}{\partial a_i \partial a_j}(\cdot)$$

and $w_j(t)$ are mutually independent Wiener processes. Taking the expectation of both sides of Eq. (8), the differential equation governing the expected value of ξ is

$$\frac{d}{dt} E[\xi] = L(E|\xi|)$$

since the $w_j(t)$ have zero expectation. Thus, setting $\xi = a_r^2$ linear, differential equations are obtained that govern second moments of the amplitudes as

$$\frac{d\bar{M}_r}{dt} = \varepsilon^2 \, 2m_{a_r} a_r + \varepsilon^2 [\sigma\sigma^T]_{a_r a_r} = \varepsilon^2 \sum_{s=1}^{M} A_{rs} \bar{M}_s \tag{10}$$

where the coefficient A_{rs} are given by

$$A_{rr} = \frac{1}{2} \kappa_{1,rr} S_{ff}(0) + \frac{1}{2} \kappa_{2,rr} S_{ff}(2\omega_r)$$

$$+ \frac{1}{4} \sum_{s \neq r}^{M} [\kappa_{5,rs} S_{ff}(\Omega^+_{rs}) - \kappa_{6,rs} S_{ff}(\Omega^-_{rs})$$

$$+ \kappa_{7,rs} \psi_{ff}(\Omega^-_{rs}) - \kappa_{8,rs} \psi_{ff}(\Omega^+_{rs})]$$

$$+ \frac{1}{4} \sum_{s=1}^{N-M} [\kappa_{9,rs} \bar{S}_{ff}(\Omega^+_{r,M+s}) - \kappa_{10,rs} \bar{S}_{ff}(\Omega^-_{r,M+s})$$

$$+ \kappa_{11,rs} \bar{\psi}_{ff}(\Omega^-_{r,M+s}) - \kappa_{12,rs} \bar{\psi}_{ff}(\Omega^+_{r,M+s})]$$

$$+ \frac{1}{4} \sum_{s=1}^{m} [\kappa_{13,rs} \hat{S}_{ff}(\omega_r) - \kappa_{14,rs} \hat{\psi}_{ff}(\omega_r)]$$

$$+ 2\Delta\delta_r \tag{11}$$

$$A_{rs} = \frac{1}{4} [\kappa_{2,rs} S_{ff}(\Omega^+_{rs}) + \kappa_{4,rs} S_{ff}(\Omega^-_{rs})] \text{ , } r \neq s; \quad r,s = 1,2,\ldots M \tag{12}$$

where the quantities κ_i, $i=1,2,\ldots 14$ are defined in Appendix D.

Thus, from Eq. (10), necessary and sufficient conditions for stability in the second moment are that all eigenvalues of the matrix $A = [A_{ij}]$ have negative real parts. These conditions may be found by applying the Routh-Hurwitz criteria to matrix A.

If the excitation has a broad-band spectrum with a constant spectral density S_0, over a wide band of frequencies, i.e. $S_{ff}(\omega) = \bar{S}_{ff}(\omega) = \hat{S}_{ff}(\omega) = S_0$ and $\psi_{ff}(\omega) = \bar{\psi}_{ff}(\omega) = \hat{\psi}_{ff}(\omega) = 0$, then the matrix A remains a full (no zero elements) nxn matrix. For multi-degree of freedom systems, the Routh-Hurwitz conditions involve computation of determinants of large matrices, which is tedious. Thus, the stability conditions for M=2 can be written as

$$A_{11} + A_{22} < 0$$

$$A_{11}A_{22} - A_{12}A_{21} < 0 \tag{13}$$

It is evident from Eqs. (12) that $\kappa_{2,rs}$ and $\kappa_{4,rs}$ are positive quantities, implies $A_{12}A_{21} > 0$, thus, the stability conditions of Eq. (13) can be simplified to

$$\begin{aligned}
&2\Delta\delta_1 + S_o\alpha_{11} < 0 \\
&2\Delta\delta_2 + S_o\alpha_{22} < 0 \\
&S_o^{\,2}(\alpha_{11}\alpha_{22} - \beta) + 2S_o(\Delta\delta_1\alpha_{22} + \Delta\delta_2\alpha_{11}) + 4\Delta\delta_1\Delta\delta_2 > 0
\end{aligned} \tag{14}$$

where

$$\alpha_{11} = \frac{1}{2}(\kappa_{1,11} + \kappa_{2,11}) + \frac{1}{4}(\kappa_{5,12} - \kappa_{6,12})$$

$$+ \frac{1}{4}\sum_{s=1}^{N-M}(\kappa_{9,1s} - \kappa_{10,1s}) + \frac{1}{4}\sum_{s=1}^{m}\kappa_{13,1s}$$

$$\alpha_{22} = \frac{1}{2}(\kappa_{1,22} + \kappa_{2,22}) + \frac{1}{4}(\kappa_{5,21} - \kappa_{6,21})$$

$$+ \frac{1}{4}\sum_{s=1}^{N-M}(\kappa_{9,2s} - \kappa_{10,2s}) + \frac{1}{4}\sum_{s=1}^{m}\kappa_{13,2s}$$

$$\beta = \frac{1}{16}(\kappa_{2,12} + \kappa_{4,12})(\kappa_{2,21} + \kappa_{4,21})$$

Returning now to the M-degree of freedom system, some particular forms of excitation spectrum $S(\omega)$ are considered, whose values are small everywhere except in the neighborhood of some ω_o; i.e. $S(\omega)$ vanishes outside the bandwidth $\omega_o - 1/2\,\Delta\omega_o < \omega < \omega_o + 1/2\,\Delta\omega_o$. The correlation time of such a stochastic process is $0(1/\Delta\omega_o)$, while the relaxation time of the amplitude process is $0(1/\epsilon^2)$. Therefore, if $\Delta\omega_o >> \epsilon^2$, the Markov approximation obtained by use of the limit theorem will remain valid. In the following,

cases in which $\omega_o = 2\omega_r$ and $\omega_o = |\omega_s \pm \omega_r|$, r,s = 1,2...M are considered. Although dealing with a multi-degree of freedom system, knowledge of the frequency content of the excitation can simplify stability analysis.

Taking $\omega_o = |\omega_r \pm \omega_s|$, $(r \neq s)$, it is evident that off-diagonal elements of the matrix A except A_{rs} and A_{sr} are identically zero, i.e. $A_{ij} = 0$, $i,j \neq r,s$, $i \neq j$ and $A_{ii} = 2\Delta\delta_i$, $i \neq r,s$. Furthermore, if $\omega_o = 2\omega_r$ then all the off-diagonal elements of matrix A are identically zero; i.e. $A_{ij} = 0$, $i \neq j$, and $A_{ii} = 2\Delta\delta_i$, $i \neq r,s$ stability conditions for $\omega_o = \omega_s + \omega_r$ can be written as

$$\Delta\delta_r + \frac{1}{8}\kappa_{5,rs}\, S_{ff}(\Omega^+_{rs}) < 0$$

$$\Delta\delta_s + \frac{1}{8}\kappa_{5,sr}\, S_{ff}(\Omega^+_{rs}) < 0 \qquad (15)$$

$$\frac{1}{64} S^2_{ff}(\Omega^+_{rs})(\kappa_{5,rs}\kappa_{5,sr} - \kappa_{2,rs}\kappa_{2,sr})$$

$$+ \frac{1}{8} S_{ff}(\Omega^+_{rs})(\Delta\delta_r\kappa_{5,sr} + \Delta\delta_s\kappa_{5,rs}) + \Delta\delta_r\Delta\delta_s > 0$$

Similarly, if $\omega_o = |\omega_r - \omega_s|$, $r \neq s$, the corresponding stability conditions are

$$\Delta\delta_r - \frac{1}{8}\kappa_{6,rs}\, S_{ff}(\Omega^-_{rs}) < 0$$

$$\Delta\delta_s - \frac{1}{8}\kappa_{6,sr}\, S_{ff}(\Omega^-_{rs}) < 0 \qquad (16)$$

$$\frac{1}{64} S^2_{ff}(\Omega^-_{rs})(\kappa_{6,rs}\kappa_{6,sr} - \kappa_{4,rs}\kappa_{4,sr})$$

$$+ \frac{1}{8} S_{ff}(\Omega^-_{rs})(\Delta\delta_r\kappa_{4,sr} + \Delta\delta_s\kappa_{4,rs}) + \Delta\delta_r\Delta\delta_s > 0$$

Finally, taking $\omega_o = 2\omega_r$, the stability condition is obtained as

$$\Delta\delta_r + \frac{1}{4}\kappa_{2,rr}S_{ff}(2\omega_r) < 0 \qquad (17)$$

It may be noted that the stability condition of Eq. (17) can also be obtained from the inequalities of Eq. (15), since $\kappa_{5,rr} = \kappa_{2,rr}$.

V. Application: A Cantilever Column Subject to Stochastic Follower Force

In order to illustrate the general results obtained previously, we consider a cantilever column of length ℓ, mass per unit length m, flexural rigidity EI, and with a stochastically varying follower force $p(t) = p_o + \epsilon f(t)$ as shown in Fig. 1. Equation of motion and the boundary conditions are shown as [13]

$$E^*I\frac{\partial^5 y}{\partial x^4\partial t} + EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} + p\frac{\partial^2 y}{\partial x^2} + c\frac{\partial y}{\partial t} = 0 ,$$

$$y(o,t) = \frac{\partial y(o,t)}{\partial x} = 0 ,$$

$$\frac{\partial^2 y(\ell,t)}{\partial x^2} = \frac{\partial^3 y(\ell,t)}{\partial x^3} = 0 ,$$

where E^* is coefficient of internal dissipation which is assumed to be Kelvin-Voigt type and c is the coefficient of external damping. Now, defining the dimensionless quantities

$$\xi = \frac{x}{\ell} , \quad \eta = \frac{y}{\ell} , \quad \tau = \left(\frac{EI}{m}\right)^{1/2}\frac{t}{\ell^2} , \quad \alpha = \left(\frac{I}{Em}\right)^{1/2}\frac{E^*}{\ell^2}$$

$$\beta = \frac{c\ell^2}{(EIm)^{1/2}} , \quad P = \frac{p\ell}{EI} = P_o + \epsilon F(\tau) .$$

The dimensionless equation of motion is obtained as

$$\alpha \frac{\partial^5 \eta}{\partial \xi^4 \partial \tau} + \frac{\partial^4 \eta}{\partial \xi^4} + P \frac{\partial^2 \eta}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial \tau^2} + \beta \frac{\partial \eta}{\partial \tau} = 0 \qquad (18)$$

and the dimensionless boundary conditions are

$$\eta = \frac{\partial \eta}{\partial \xi} = 0 \qquad \text{at} \quad \xi = 0 \ ,$$

$$\frac{\partial^2 \eta}{\partial \xi^2} = \frac{\partial^3 \eta}{\partial \xi^3} = 0 \qquad \text{at} \quad \xi = 1 \ .$$

This problem was considered in a different context by Wiens and Sinha [14] and Parthasarathy and Evan-Iwanowski [15]. The discrete equations of motion corresponding to Eq. (18) are obtained by using a two term Ritz-Galerkin approximation as

$$I \ddot{g} + B \dot{g} + C g = \varepsilon D_1 F(\tau) \ g$$

where the matrices B, C and D_1 can be written explicitly as

$$B = \begin{bmatrix} 12.36\alpha+\beta & 0 \\ 0 & 485.52\alpha + \beta \end{bmatrix}, \quad C = \begin{bmatrix} 12.36+0.86P_o & -11.74P_o \\ 1.87P_o & 485.52-13.29P_o \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -0.86 & 11.74 \\ 1.87 & 13.29 \end{bmatrix} .$$

The characteristic equation of the system (Eq. (2)) is obtained as

$$\lambda^4 + (497.881\alpha+2\beta)\lambda^3 + (497.881+\alpha^2 + 497.88\ \alpha\beta + 6002.16\beta^2$$

$$- 12.436P_o)\lambda^2 + (12004.3\alpha + 497.881\beta + 252.345\alpha P_o$$

$$- 12.436\beta\ P_o)\lambda + 6002.16 + 10.595\ P_o^2 = 0 \ , \qquad (19)$$

and from Routh-Hurwitz criteria, the governing equations for stability are

$$\alpha > 0\ , \quad \beta > 0\ ,$$

$$(112.274\beta^2 + 55898.9\alpha\beta - 4.25255\ 10^6\alpha^2)P_o^2 - (7.541\ 10^8\alpha^4$$

$$- 2.842\ 10^7\alpha^3\beta + 2.855\ 10^6\alpha^2\beta^2 + 8.039\ 10^7\alpha^2$$

$$+ 18070.3\alpha\beta^3 + 6.668\ 10^6\alpha\beta + 24.872\beta^4 + 13392.7\beta^2)P_o$$

$$+ 3.587\ 10^{10}\alpha^4 + 4.608\ 10^9\alpha^3\beta + 1.473\ 10^8\alpha^2\beta^2$$

$$+ 1.344\ 10^9\alpha^2 + 767666\alpha\beta^3 + 1.115\ 10^8\alpha\beta + 995.762\beta^4$$

$$+ 223877\beta^2 > 0 \qquad (20)$$

In an earlier study of the deterministic problem, Leipholz [13] concluded that rods subjected to external viscous damping does not affect stability and can, therefore, be neglected. From Eq. (20), the critical force in Fig. 2 is not affected by the external damping for the case of internal damping $\alpha = 0$. Leipholz also showed that for the clamped-free rod, internal damping must always be considered and has a destabilized effect. In Fig. 3, the structure with internal damping, the smallest flutter load turned out to be $P_{cr} = 10.68$, which is less than $P_{cr} = 20.1$ in Fig. 2. Consequently, in this paper we shall only consider the system with internal damping (i.e . $\alpha \neq 0$, $\beta = 0$).

By using the transformation matrix $\bar{C}$ and D mentioned previously, one can show that the system has one pair of purely imaginary eigenvalues at the stability boundary in Fig. 3 and bring the system in the same form as Eq. (4). Now, the results of the last section can be readily applied.

Numerical calculations were done for different internal damping α. It is evident from Eq. (14) that the stability condition for the second moment of the linear system can be written as

$$\Delta\delta_1 + S_o\left[\frac{1}{4}\left(\kappa_{1,11} + \kappa_{2,11}\right) + \frac{1}{8}\left(\kappa_{9,11} - \kappa_{10,11}\right)\right] < 0 ,$$

the numerical results obtained are plotted in Fig. 4 which also shows the effect of internal damping on the stochastic stability boundaries of a cantilever beam with follower force. It can be seen that the stability regions reduce with decreasing value of internal damping α. Furthermore, as α approaches 0, the whole region in the parameter space $(\Delta P, S_o)$, where $\Delta P = (P_{cr} - P)/P_{cr}$, becomes unstable.

VI. Conclusion

An analytical method has been presented here for studying the stability of linear nonconservative multidegrees of freedom system subjected to stochastically varying excitation of small intensity. These systems are typically encountered in the study of the dynamic stability of elastic structures under random loads.

The equations of motion were first transformed into 2n first-order equations in the amplitude and the phase variables. These quantities, by using the method of modified stochastic averaging, under suitable conditions, converge to a Markov vector which satisfies Itô equations. From the Itô equations, criteria for mean square stability were obtained, with the aid of the Routh-Hurwitz criteria. In analogy with the deterministic results, it is found that only those values of the excitation spectrum near twice the system's natural frequencies and the sums and differences of the natural frequencies influence the stability. As an application, the stochastic stability of cantilever columns with stochastic follower force was considered.

Appendix A1

The matrices K^o, L, M, N and P are explicitly calculated using the eigenvectors and the matrix Q.

K^o, M^1, N^1 and L^1 are the partition of $[\alpha_{ij}]$

$$K^o = [\alpha_{ij}]_{2M\times2M} \quad , \begin{cases} i = 1,2,\ldots M \\ j = 1,2,\ldots M \end{cases}$$

$$M^1 = [\alpha_{ij}]_{2M\times2(N-M)} \quad , \begin{cases} i = 1,2,\ldots M \\ j = M+1,M+2,\ldots N \end{cases}$$

$$N^1 = [\alpha_{ij}]_{2(N-M)\times2M} \quad , \begin{cases} i = M+1,M+2,\ldots N \\ j = 1,2,\ldots M \end{cases}$$

$$L^1 = [\alpha_{ij}]_{2(N-M)\times2(N-M)} \quad . \begin{cases} i = M+1,M+2,\ldots N \\ j = M+1,M+2,\ldots N \end{cases}$$

M^2 and L^2 are the partition of $[\beta_{ij}]$

$$M^2 = [\beta_{ij}]_{2M\times m} \quad , \begin{cases} i = 1,2,\ldots M \\ j = 1,2,\ldots m \end{cases}$$

$$L^2 = [\beta_{ij}]_{2(N-M)\times m} \quad . \begin{cases} i = M+1,M+2,\ldots N \\ j = 1,2,\ldots m \end{cases}$$

N^2 and P^1 are the partition of $[\gamma_{ij}]$

$$N^2 = [\gamma_{ij}]_{m\times2M} \quad , \begin{cases} i = 1,2\ldots m \\ j = 1,2\ldots M \end{cases}$$

$$P^1 = [\gamma_{ij}]_{m\times2(N-M)} \quad . \begin{cases} i = 1,2,\ldots m \\ j = M+1,M+2,\ldots N \end{cases}$$

$$P^2 = [\rho_{ij}]_{m\times m} \quad . \begin{cases} i = 1,2,\ldots m \\ j = 1,2,\ldots m \end{cases}$$

where

$$D^T Q \,\overline{C} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ \gamma_{ij} & \rho_{ij} \end{bmatrix}$$

$$\alpha_{ij} = 2 \begin{bmatrix} (\underline{e}^i)^T Q\, \underline{c}^j & (\underline{e}^i)^T Q\, \underline{d}^j \\ (-\underline{f}^i)^T Q\, \underline{c}^j & (-\underline{f}^i) Q\, \underline{d}^j \end{bmatrix}_{2x2} , \quad \beta_{ij} = \begin{bmatrix} (\underline{e}^i)^T Q\, \underline{a}^j \\ (-\underline{f}^i)^T Q\, \underline{a}^j \end{bmatrix}_{2x1} ,$$

$$\gamma_{ij} = 2\, [(\underline{b}^i)^T Q\, \underline{c}^j \quad (\underline{b}^i)^T Q\, \underline{d}^j]_{1x2} , \quad \rho_{ij} = [(\underline{b}^i)^T Q\, \underline{a}^j]_{1x1} .$$

Appendix -

The matrices $X_a^{(1)}$, $X_\phi^{(1)}$ and $X_W^{(1)}$ are defined as follow

$$X_{a_j}^{(0)}(a,\phi,t) = \frac{1}{2} a_j[J_{1,jj}^- + J_{1,jj}^+ \cos 2\phi_j + H_{1,jj}^+ \sin 2\phi_j]$$

$$+ \frac{1}{2} \sum_{k\neq j}^{M} a_k[J_{1,jk}^+ \cos(\phi_j - \phi_k) + J_{1,jk}^- \cos(\phi_j - \phi_k)$$

$$+ H_{1,jk}^- \sin(\phi_j - \phi_k) + H_{1,jk}^+ \sin(\phi_j + \phi_k)]$$

$$X_{a_j}^{(1)}(Z,\phi,t) = \frac{1}{2} \sum_{k=1}^{N-M} Z_{2k-1}(J_{2,jk}^+ \sin\hat{\phi}_j^- - J_{2,jk}^- \sin\hat{\phi}_j^+$$

$$- H_{2,jk}^- \cos\hat{\phi}_j^- + H_{2,jk}^+ \cos\hat{\phi}_j^+)$$

$$+ \frac{1}{2} \sum_{k=1}^{N-M} Z_{2k}(H_{2,jk}^- \sin\hat{\phi}_j^- + H_{2,jk}^+ \sin\hat{\phi}_j^+$$

$$+ J_{2,jk}^+ \cos\hat{\phi}_j^- + J_{2,jk}^- \cos\hat{\phi}_j^+)$$

$$X_{a_j}^{(2)}(W,\phi,t) = \frac{1}{2} \sum_{k=1}^{m} W_k(M_{2j-1,k}^2 \sin\phi_j + M_{2j,k}^2 \cos\phi_j)$$

$$X_{\phi_j}^{(o)}(a,\phi,t) = \frac{1}{2} a_j(H_{1,jj}^- + H_{1,jj}^+ \cos 2\phi_j - J_{1,jj}^- \sin 2\phi_j)$$

$$+ \sum_{k\neq j}^{M} \frac{a_k}{2} [H_{1,jk}^- \cos(\phi_j - \phi_k) + H_{1,jk}^+ \cos(\phi_j + \phi_k)$$

$$- J_{1,jk}^+ \sin(\phi_j - \phi_k) - J_{1,jk}^- \sin(\phi_j + \phi_k)]$$

$$X_{\phi_j}^{(1)}(Z,\phi,t) = \frac{1}{2} \sum_{k=1}^{N-M} Z_{2k-1}(J_{2,jk}^+ \cos\hat{\phi}_j^- - J_{2,jk}^- \cos\hat{\phi}_j^+$$

$$+ H^{+}_{2,jk}\sin\hat{\Phi}^{-}_{j} - H^{+}_{2,jk}\sin\hat{\Phi}^{+}_{j})$$

$$+ \frac{1}{2}\sum_{k=1}^{N-M} Z_{2k}(H^{-}_{2,jk}\cos\hat{\Phi}^{-}_{j} + H^{+}_{2,jk}\cos\hat{\Phi}^{+}_{j}$$

$$- J^{+}_{2,jk}\sin\hat{\Phi}^{-}_{j} - J^{-}_{2,jk}\sin\hat{\Phi}^{+}_{j})$$

$$X^{(2)}_{\phi_j}(W,\phi,t) = \sum_{k=1}^{m} W_k(M^2_{2j-1,k}\cos\Phi_j - M^2_{2j,k}\sin\Phi_j)$$

$$X^{(o)}_{Z_{2r-1}}(a,\phi,t) = \sum_{s=1}^{M} \frac{a_s}{2}(H^{-}_{3,rs}\cos\hat{\Phi}^{-}_{s} + H^{+}_{3,rs}\cos\hat{\Phi}^{+}_{s}$$

$$+ J^{+}_{3,rs}\sin\hat{\Phi}^{-}_{s} - J^{+}_{3,rs}\sin\hat{\Phi}^{+}_{s})$$

$$X^{(o)}_{Z_{2r}}(a,\phi,t) = \sum_{s=1}^{M} \frac{a_s}{2}(H^{-}_{3,rs}\sin\hat{\Phi}^{-}_{r} + H^{+}_{3,rs}\sin\hat{\Phi}^{+}_{r}$$

$$+ J^{+}_{3,rs}\cos\hat{\Phi}^{-}_{s} + J^{-}_{3,rs}\cos\hat{\Phi}^{+}_{s})$$

$$X^{(0)}_{W_i} = \sum_{\ell=1}^{M} a_\ell\,(N^2_{1,2\ell-1}\sin\Phi_\ell + N^2_{1,2\ell}\cos\Phi_\ell)$$

where

$$\hat{\Phi}^{\pm}_{j} = (\omega_j \pm \omega_{M+k})t + \phi_j\ ,$$

$$\hat{\Phi}^{\pm}_{s} = (\omega_s \pm \omega_{M+r})t + \phi_s\ .$$

Appendix B

The drift and diffusion terms can be obtained by applying the following limits directly to equation (7).

$$m_{a_j} = \underset{t}{M} \{ \int_{-\infty}^{0} [\sum_{i=1}^{M} < \frac{\partial F_j^1}{\partial a_i} F_{i,\tau}^1 + \frac{\partial F_j^1}{\partial \phi_i} F_{i,\tau}^2 > R_{ff}(\tau)$$

$$+ \sum_{i=1}^{M} < \frac{\partial F_j^1}{\partial a_i} G_{i,\tau}^1 + \frac{\partial F_j^1}{\partial \phi_i} G_{i,\tau}^2 + \frac{\partial G_j^1}{\partial \phi_i} F_{i,\tau}^2 > R_{fg}(\tau)$$

$$+ \sum_{i=1}^{M} < \frac{\partial G_j^1}{\partial \phi_i} G_{i,\tau}^2 > R_{gg}(\tau)$$

$$+ \sum_{r=1}^{N-M} < \frac{\partial F_j^1}{\partial z_{2r-1}} F_{2r-1,\tau}^3 + \frac{\partial F_j^1}{\partial z_{2r}} F_{2r,\tau}^3 > \Big|_{z=0,w=0} R_{ff} e^{\delta_r \tau}$$

$$+ \sum_{r=1}^{N-M} < \frac{\partial F_j^1}{\partial z_{2r-1}} G_{2r-1,\tau}^3 + \frac{\partial F_j^1}{\partial z_{2r}} G_{2r,\tau}^3 > \Big|_{z=0,w=0} R_{fg}(\tau) e^{\delta_r \tau}$$

$$+ \sum_{k=1}^{m} < \frac{\partial F_j^1}{\partial w_k} F_{k,\tau}^4 > \Big|_{z=0,w=0} R_{ff}(\tau) e^{\alpha_k \tau}$$

$$+ \sum_{k=1}^{m} < \frac{\partial F_j^1}{\partial w_k} G_{k,\tau}^4 > \Big|_{z=0,w=0} R_{fg}(\tau) e^{\alpha_k \tau}] \, d\tau \} + \underset{t}{M} \{ F_j^0 \} \; ,$$

$$m_{\phi_j} = \underset{t}{M} \{ \int_{-\infty}^{0} [\sum_{i=1}^{M} < \frac{\partial F_j^2}{\partial \phi_i} F_{i,\tau}^2 + \frac{\partial F_j^2}{\partial a_i} F_{i,\tau}^1 > R_{ff}(\tau)$$

$$+ \sum_{i=1}^{M} < \frac{\partial F_j^2}{\partial \phi_i} G_{i,\tau}^2 + \frac{\partial F_j^2}{\partial a_i} G_{i,\tau}^1 + \frac{\partial G_j^2}{\partial \phi_i} F_{i,\tau}^2 + \frac{\partial G_j^2}{\partial a_i} F_{i,\tau}^1 > R_{fg}(\tau)$$

$$+ \sum_{i=1}^{M} \quad \frac{\partial G_j^2}{\partial \phi_i} G_{i,\tau}^2 + \frac{\partial G_j^2}{\partial a_i} G_{i,\tau}^1 > R_{gg}(\tau)$$

$$+ \sum_{r=1}^{N-M} < \frac{\partial F_j^2}{\partial z_{2r-1}} F_{2r-1,\tau}^3 + \frac{\partial F_j^2}{\partial z_{2r}} F_{2r,\tau}^3 > \Big|_{z=0,w=0} R_{ff}(\tau) e^{\delta_r \tau}$$

$$+ \sum_{r=1}^{N-M} < \frac{\partial F_j^2}{\partial z_{2r-1}} G_{2r-1,\tau}^3 + \frac{\partial F_j^2}{\partial z_{2r}} G_{2r,\tau}^3 > \Big|_{z=0,w=0} R_{fg}(\tau) e^{\delta_r \tau}$$

$$+ \sum_{k=1}^{m} < \frac{\partial F_j^2}{\partial w_k} F_{k,\tau}^4 > \Big|_{z=0,w=0} R_{ff}(\tau) e^{\alpha_k \tau}$$

$$+ \sum_{k=1}^{m} < \frac{\partial F_j^2}{\partial w_k} G_{k,\tau}^4 > \Big|_{z=0,w=0} R_{fg}(\tau) e^{\alpha_k \tau}] d\tau \} ,$$

$$[\sigma\sigma^T]_{a_i a_j} =$$

$$\underset{t}{M} \{ \int_{-\infty}^{\infty} [<F_i^1 F_{j,\tau}^1 > R_{ff}(\tau) + <F_i^1 G_{i,\tau}^1 + G_i^1 F_{j,\tau}^1 > R_{fg}(\tau)$$

$$+ \langle G_i^1 G_{j,\tau}^1 \rangle R_{gg}(\tau)]d\tau\} ,$$

$$[\sigma\sigma^T]_{a_i\phi_j} =$$

$$\underset{t}{M}\{ \int_{-\infty}^{\infty} [\langle F_i^1 F_{j,\tau}^2 \rangle R_{ff}(\tau) + \langle F_i^1 G_{j,\tau}^2 + G_i^1 F_{j,\tau}^2 \rangle R_{fg}(\tau)$$

$$+ \langle G_i^1 G_{j,\tau}^2 \rangle R_{gg}(\tau)] d\tau\} ,$$

$$[\sigma\sigma^T]_{\phi_i\phi_j} = \underset{t}{M}\{\int_{-\infty}^{\infty}[\langle F_i^2 F_{j,\tau}^2 \rangle R_{ff}(\tau) + \langle F_i^2 G_{j,\tau}^2 + G_i^2 F_{j,\tau}^2 \rangle R_{fg}(\tau)$$

$$+ \langle G_i^2 G_{j,\tau}^2 \rangle R_{gg}(\tau)] d\tau\} ,$$

where $F_{j,\tau}^i(\cdot) = F_j^i(\cdot, t+\tau)$, $G_{j,\tau}^i(\cdot) = G_j^i(\cdot, t+\tau)$, $\underset{t}{M}$ is the averaging operator defined by

$$\underset{t}{M}(\cdot) = \lim_{T\to\infty} T^{-1}\int_0^T (\cdot)\, dt$$

$R_{ff}(t) = \langle f(t) \cdot f(t+\tau) \rangle$, $\langle \cdot \rangle$ denotes the expectation, and in evaluating the expectation in Eqs. (7), a and ϕ are treated as constants. Then (a,ϕ) in Eqs. (7) can be uniformly approximated in the weak sense by a Markov diffusion process having drift vector $\underline{m}$ and diffusion matrix $[\sigma\sigma^T]$.

Appendix A

The quantities κ_i, i=1,2,....15 are defined as follow:

$$\kappa_{1,rr} = {J^{+}_{1,rr}}^2 \qquad \kappa_{1,rs} = J^{+}_{1,rr}J^{+}_{1,ss}$$

$$\kappa_{2,rr} = {H^{+}_{1,rr}}^2 + {J^{-}_{1,rr}}^2 \qquad \kappa_{2,rs} = {H^{+}_{1,rs}}^2 + {J^{-}_{1,rs}}^2$$

$$\kappa_{3,rr} = H^{-}_{1,rr}J^{+}_{1,rr} \qquad \kappa_{3,sr} = H^{-}_{1,ss}J^{+}_{1,rr}$$

$$\kappa_{4,rs} = {H^{-}_{1,rs}}^2 + {J^{+}_{1,rs}}^2$$

$$\kappa_{5,rs} = H^{+}_{1,rs}H^{+}_{1,sr} + J^{-}_{1,rs}J^{-}_{1,sr}$$

$$\kappa_{6,rs} = H^{-}_{1,rs}H^{-}_{1,sr} - J^{+}_{1,rs}J^{+}_{1,sr}$$

$$\kappa_{7,rs} = H^{-}_{1,sr}J^{+}_{1,rs} + H^{-}_{1,rs}J^{+}_{1,sr}$$

$$\kappa_{8,rs} = H^{+}_{1,sr}J^{-}_{1,rs} - H^{+}_{1,rs}J^{-}_{1,sr}$$

$$\kappa_{9,rs} = H^{+}_{2,rs}H^{+}_{3,sr} + J^{-}_{2,rs}J^{-}_{3,sr}$$

$$\kappa_{10,rs} = H^{-}_{2,rs}H^{-}_{3,sr} - J^{+}_{2,rs}J^{+}_{3,sr}$$

$$\kappa_{11,rs} = H^{-}_{3,sr}J^{+}_{2,rs} + H^{-}_{2,rs}J^{+}_{3,sr}$$

$$\kappa_{12,rs} = H^{+}_{3,sr}J^{-}_{2,rs} - H^{+}_{2,rs}J^{-}_{3,sr}$$

$$\kappa_{13} = M^{2}_{2r-1,s}N^{2}_{s,2r-1} + M^{2}_{2r,s}N^{2}_{s,2r}$$

$$\kappa_{14} = M^{2}_{2r,s}N^{2}_{s,2r-1} - M^{2}_{2r-1,s}N^{2}_{s,2r}$$

$$\kappa_{15} = {s^{1}_{2r-1}}^{2} + {s^{1}_{2r}}^{2}$$

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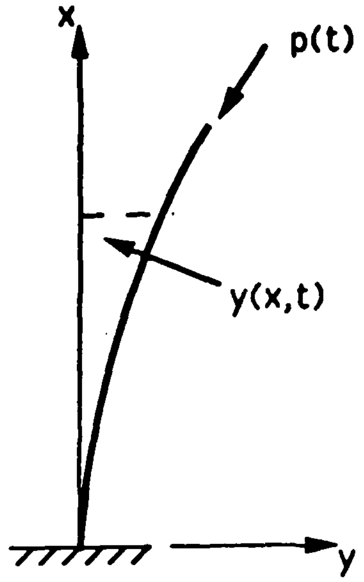

Fig. 1 Column with stochastic follower force

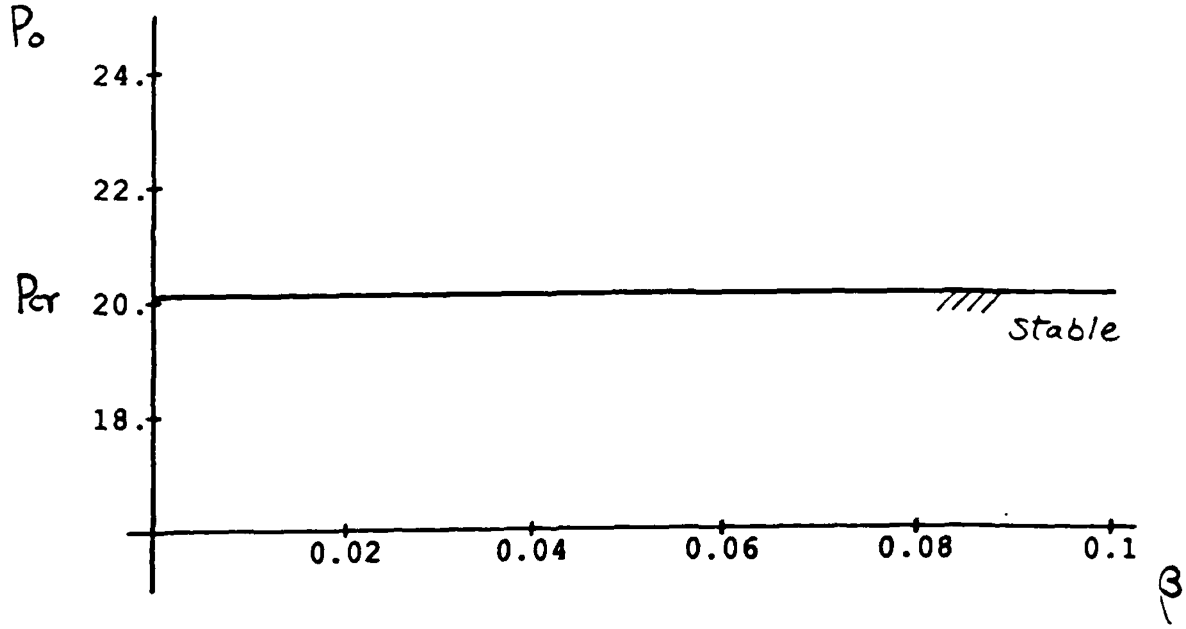

Fig. 2 Stability boundary of flutter load with external damping only

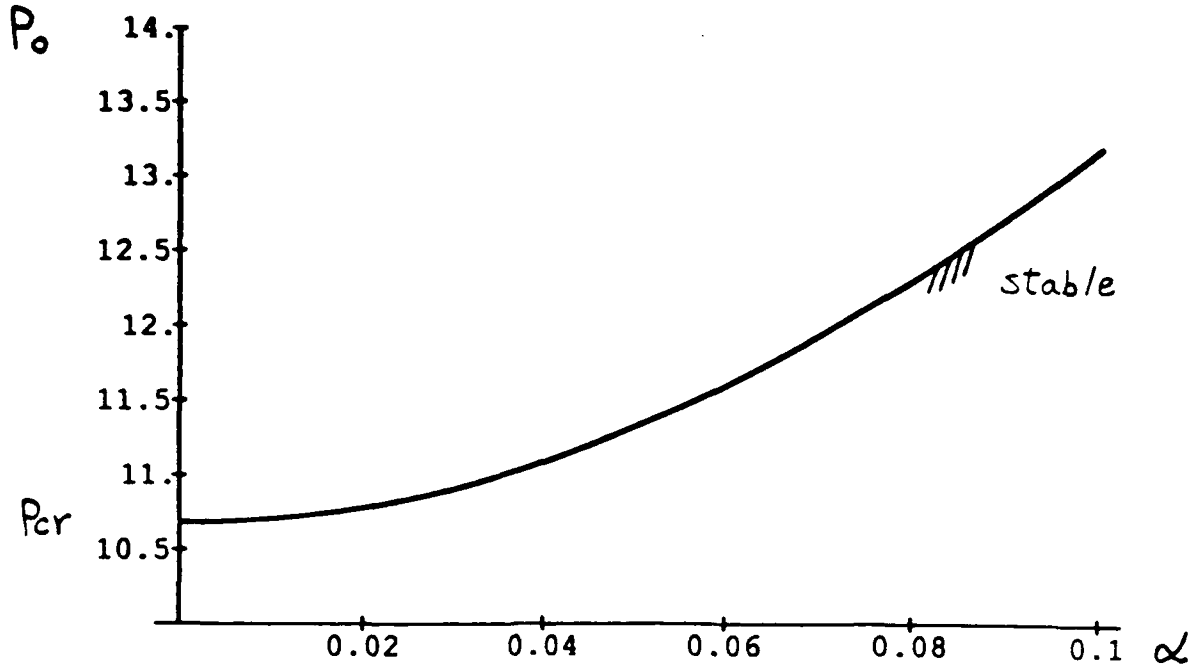

Fig. 3 Stability boundary of flutter load with internal damping only

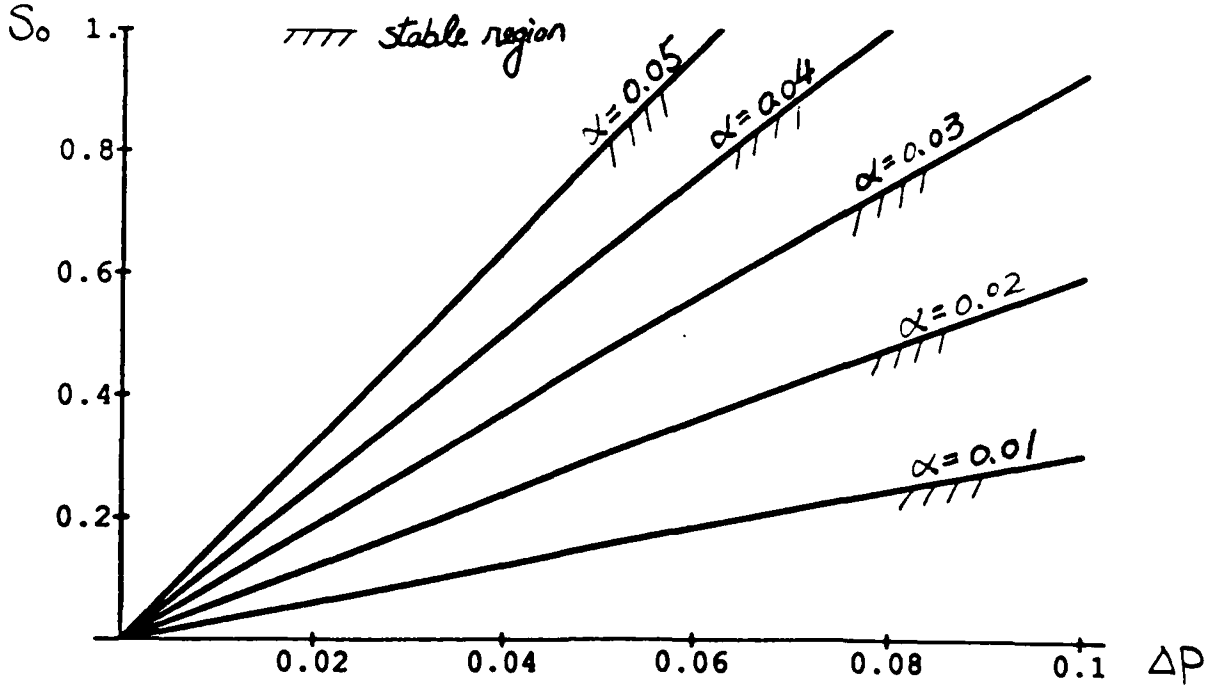

Fig. 4 Moment stability with different internal damping

APPENDIX B

Introduction

There exist several natural phenomena that vary in a random manner due to the effects of large numbers of unknown factors. Dynamic systems in such environments are subjected to stochastic excitations. The stability and the nonlinear response of such stochastic systems have become of increasing interest in engineering. Examples of stochastic excitations include forces generated by jet and rocket engines in modern high-powered aircraft, space and missile structures, as well as excitation due to earthquakes, ocean waves and wind gusts. They fluctuate randomly over a wide band of frequencies and have to be considered as stochastic functions of time defined only in probabilistic terms. The effects of such fluctuations on nonlinear systems have been studied by various researchers. For one dimensional systems, Horsthemke and Lefever (1977) and Arnold et. al. (1978) established the existence of transition or bifurcations solely induced by noise. These ideas were extended to two dimensional systems by Arnold et. al. (1979) using the well-known Lotka-Voltera model for two interacting populations. The effect of stochastic perturbations upon a dynamical system exhibiting co-dimension one bifurcations has been studied by Baras et. al. (1982) Graham (1982), Lefever and Turner

(1984) and the author (1988a,1988b, 1989). The effect of additive noise was considered in (1982), while the effect of multiplicative noise was studied by Graham (1982). The author (1988b,1989) considered a more general problem in R^n, with both multiplicative and additive stochastic excitations. Even though a large amount of work in co-dimension one stochastic bifurcations has been reported, there is still considerably more work to be done in stochastically perturbed co-dimension two bifurcations. The goal of this paper is to present the results pertaining to the statistical as well as the sample behavior of nonlinear systems that exhibit a particular co-dimension two bifurcation and subjected to random excitations.

A dynamical system undergoes a co-dimension two bifurcation due to the presence of additional degeneracies other than those encountered for the simple and the Hopf bifurcations. Such degeneracies can be classified into two types, in the first the linear part is similar to that of co-dimension one bifurcation but additional degeneracies occur in the nonlinear terms of the normal form or higher order degeneracies occur in the linear part, and in the second the linear part of the vector field is doubly degenerate. In this paper, we shall consider the stochastic version of the co-dimension two bifurcation of the latter type. There are three cases of such types of degeneracy with two, three and four dimensional center manifold where the linear part can have: 1) two zero eigenvalues, 2) a pair of pure imaginary eigenvalues and a zero eigenvalue and 3) two pairs of pure imaginary eigenvalues without 1:1 resonance, respectively. For a deterministic symmetric system these are represented by the following three cases which depend on two parameters μ_1 and μ_2:

(1) Double zero, nondiagonalizable eigenvalues

$$\ddot{x} + \mu_2\dot{x} + \dot{x}x^2 + \mu_1 x \pm x^3 = 0 . \tag{1a}$$

(2) Simple zero and pure imaginary pair of eigenvalues

$$\dot{r} = \mu_1 r + arz + (cr^3 + drz^2), \quad \dot{z} = \mu_2 + br^2 - z^2 + (er^2 z + fz^3). \tag{1b}$$

(3) Two pure imaginary pairs of eigenvalues without resonance

$$\dot{r}_1 = \mu_1 r_1 + a_{11} r_1^3 + a_{12} r_1 r_2^2 , \quad \dot{r}_2 = \mu_2 r_2 + a_{21} r_1^2 + a_{22} r_2^3 . \qquad (1c)$$

The detailed analysis of these cases with various classifications and unfoldings are summarized in Guckenheimer and Holmes (1983). For simplicity, in this paper we shall restrict our attention to the stochastic version of case 1, i.e., double zero eigenvalues with non-semisimple forms as in Graham (1987).

Statement of the Problem

Consider a co-dimension two bifurcation associated with nonsemi-simple double zero eigenvalues, whose center manifold is two dimensional and the associated normal form is given by Guckenheimer and Holmes (1983)

$$\begin{aligned} \dot{u} &= v , \\ \dot{v} &= \mu_1 u + \mu_2 v \pm u^3 - u^2 v \end{aligned} \qquad (2)$$

where μ_1 and μ_2 are the unfolding parameters and $\mu_1 = \mu_2 = 0$ define the co-dimension two singularity point. We are interested here in the case where the normal form is perturbed by weak Gaussian white noise and we assume without proof that the associated normal form is obtained by letting $\mu_1 = \mu_0\left(1 + \varepsilon^{1/2} n_1(t)\right)$ and by introducing an additive noise term $\varepsilon^{1/2} n_2(t)$ in the second equation. It is worth noting that the influence of additive noise for this case was also considered by Graham (1987) and as in his case the emphasis will be given more to the bifurcation behavior as opposed to the derivation of the stochastic normal forms. Moreover, since the normal form for this case represent the van der Pol - Duffing oscillator, the problem below, given by equation (3) can also be viewed as van der Pol - Duffing oscillator under both parametric and external excitations. Detailed analysis of the stochastic normal forms for various nonlinear stochastic systems have been given by Sri Namachchivaya (1989). Thus consider

$$\begin{aligned} \dot{u} &= v , \\ \dot{v} &= \mu_0 u + \mu_2 v \pm u^3 - u^2 v + \varepsilon^{1/2}\left[\mu_0 u n_1(t) + n_2(t)\right] . \end{aligned} \qquad (3)$$

For the linear equation, since the stability is not affected by the non-homogeneous terms, the following set of Itô equations is examined:

$$\dot{x} = Bx + \sigma x \, dw$$

where

$$x = (u,v) \ , \quad B = \begin{bmatrix} 0 & 1 \\ \mu_0 & \mu_2 \end{bmatrix} \ , \quad \sigma = \begin{bmatrix} 0 & 0 \\ \nu & 0 \end{bmatrix}$$

and the excitation term is $\mu_0 n(t)dt = \nu dw$. The almost sure asymptotic stability condition is obtained, using the approach of Khasminskii (1967), by examining the Lyapunov exponent λ. For the above equation, the value of λ can be obtained in the same manner as in Kozin and Prodromou (1971) and Nishioka (1976) and is explicitly given as

$$\lambda = \int_0^{2\pi} Q(\theta)p(\theta) \, d\theta$$

where

$$Q(\theta) = (\mu_0 + 1) \cos\theta \sin\theta + \mu_2 \sin^2\theta + \frac{1}{2} \nu^2 \cos^2\theta \cos 2\theta$$

$$p(\theta) = \begin{cases} \dfrac{C \, S(-\pi/2,\theta]}{\nu^2 \cos^4\theta \, s(\theta)} & -\pi/2 \le \theta < \pi/2 \\ p(\theta-\pi) & \pi/2 \le \theta \le 3\pi/2 \end{cases}$$

$$S(-\pi/2,\theta) = \int_{-\pi/2}^{\theta} s(\xi)d\xi \ , \quad s(\xi) = \frac{1}{\cos^2\xi} \exp\left[\frac{\tan\xi}{3\nu^2}(f(\xi))\right]$$

$$f(\xi) = 2\tan^2\xi - 3\mu_2 \tan\xi - 6\mu_0$$

and C is a normalizing constant. If λ is negative, the sample paths are stable with probability one. However, in this paper our attention will be focused on the analysis of the nonlinear system.

Analysis

It is obvious that for both $\pm u^3$, the two dimensional system undergoes stochastically perturbed co-dimension one bifurcations when the parameters μ_0 and μ_2 take the values $\mu_2 \neq 0$, $\mu_0 = 0$ (simple) and $\mu_0 < 0$ and $\mu_2 = 0$ (Hopf) and have been studied before (Arnold et al., 1978; Sri Namachchivaya, 1988). Thus, in this section, we consider the bifurcations associated with the reduced normal form which represents a weakly perturbed conservative system. Such a reduction is obtained by using the rescalings

$$\mu_0 = \varepsilon^2 \nu_0, \ \mu_2 = \varepsilon^2 \nu_2, \ u = \varepsilon \bar{u}, \ v = \varepsilon^2 \bar{v}, \ \bar{t} = \varepsilon t ,$$

and henceforth omitting the bars from the scaled variables for simplicity, we can reduce Eq. (3) to the form

$$\frac{du}{dt} = v .$$

$$\frac{dv}{dt} = \nu_0 u \pm u^3 + \varepsilon[(\nu_2 - u^2)v] + \varepsilon^{1/2}[\nu_0 u n_1(t) + n_2(t)], \qquad (4)$$

which can be interpreted as either a Stratonovich or an Ito equation since the correction term is identically zero. Now introducing $H = v^2/2 + P_{1,2}(u)$, $P_{1,2}(u) = -\nu_0 u^2/2 \pm u^4/4$, $G(u) = (\nu_2 - u^2)$, the Ito equations for u and H can be written as

$$du = \sqrt{Q_{1,2}(u)}\, dt$$

$$dH = \varepsilon\{Q_{1,2}(u)G(u) + F(u)\}dt + \varepsilon^{1/2}\sqrt{Q_{1,2}(u)}\, \{\sigma_r^i(u)\, dw_i\} \qquad (5)$$

where

$$Q_{1,2}(u) = 2(H - P_{1,2}(u)) , \ F(u) = \frac{1}{2}\sum_{i=1}^{2}(\sigma_v^i(u))^2, \ \sigma_v^1 = \nu_0 u, \ \sigma_v^2 = 1.$$

In the sequel we consider only two cases, namely:

(i) $H = v^2/2 + p_1(u)$, $\nu_o < 0$, and the fixed points are given by $(0,0) \leftrightarrow$ stable and $(\pm\sqrt{-\nu_0}, 0) \leftrightarrow$ unstable. The Hamiltonian levels of interest lie within $H = (0, \nu_o^2/4)$ and $u_0^{\pm} = \pm[-\nu_o + (\nu_o^2 - 4H)^{1/2}]^{1/2}$.

(ii) $H = v^2/2 + P_2(u)$, $\nu_o > 0$, and the fixed points are given by $(0,0) \leftrightarrow$ unstable and $(\pm\sqrt{\nu_o}, 0) \leftrightarrow$ stable. The Hamiltonian levels of interest lie within $H = (-\nu_o^2/4, 0)$ and $u_0^{\pm} = [\nu_o \pm (\nu_o^2 + 4H)^{1/2}]^{1/2}$.

Now applying the theorem of Khasminskii (1968), we obtain the one dimensional Itô equation

$$dH = \bar{A}(H)\, dt + \bar{\sigma}_{HH}(H)\, dw . \tag{6}$$

where

$$\bar{A}(H) = \left(\frac{1}{\Lambda(H)}\right) \int_{u_o^-(H)}^{u_o^+(H)} \left\{G(u)\sqrt{Q_{1,2}(u)} + \frac{F(u)}{\sqrt{Q_{1,2}(u)}}\right\} du = \frac{\{B(H) + C(H)\}}{\Lambda(H)}$$

$$B(H) = \left(\frac{2}{15}\right)\left(\frac{\nu_o}{2-m}\right)^{5/2} \{[5(\nu_2/\nu_o)(2-m)^2 - 4(m^2 - m+1)]E(m)$$

$$- 2[5(\nu_2/\nu_o)(2-m)(1-m) - (2-m)(1-m)]F(m)\}$$

$$C(H) = \left(\frac{\nu_o}{2-m}\right)^{1/2} [(S_{11}\nu_o^2)E(m) + (S_{22}/2\nu_o)(2-m)F(m)]$$

$$\bar{\sigma}^2_{HH}(H) = \left(\frac{1}{\Lambda(H)}\right) \int_{u_o^-(H)}^{u_o^+(H)} \sqrt{Q_{1,2}(u)}\ \{2F(u)\} du = \frac{\sigma^2_{HH}}{\Lambda(H)} ,$$

$$\sigma^2_{HH}(H) = \left(\frac{2}{15}\right)\left(\frac{\nu_o}{2-m}\right)^{5/2} \{[5(S_{22}/\nu_o)(2-m)^2 + 4S_{11}\nu_o^{\,2}(m^2-m+1)]\ E(m)$$

$$- 2[5(S_{22}/\nu_o)(2-m)(1-m) + S_{11}\nu_o^2(2-m)(1-m)]F(m)\}$$

$$\Lambda(H) = \int_{u_o^-(H)}^{u_o^+(H)} \frac{du}{\sqrt{Q_{1,2}(u)}} = \frac{(2-m)^{1/2}}{\sqrt{\nu_o}} F(m) \;,\; T(H) = 2\Lambda(H)$$

$$H = \frac{\nu_o^2(m-1)}{(2-m)^2} \;,\quad \text{and} \quad 2C(H) = \frac{d}{dH}[\sigma_{HH}^2]$$

In the above equations F(m) and E(m) are complete elliptic integrals of the first and second kinds, respectively. By solving the corresponding Fokker-Planck equation the stationary probability density is obtained as

$$w_{st}(H) = \text{const.} \quad \exp\{\psi\} \qquad (7)$$

where

$$\psi = 2\int \frac{B(H)}{\sigma_{HH}^2(H)} dH + \ln(\Lambda(H)) \;.$$

The stationary probability density functions are shown in Fig. 1 for various values of excitations. In Fig. 1a the parametric excitation is varied while keeping the external excitation fixed and vice versa in Fig. 1b.

Following the arguments of Stratonovich (1963), the probability density that the displacement u at time t given the value of H, i.e. $p(u,t|H)$, is proportional to the time a system spent at u knowing that the energy level is H. Furthermore, the time u(t) spends at the point u is inversely proportional to the velocity and thus

$$w(u|H) = \frac{1}{T(H)} \frac{1}{\sqrt{Q_{1,2}(u)}} \quad \text{and} \quad w(u,H) = w(H)\; W(u|H)$$

and the density in u and v can be written as

$$w(u,v) = \text{const.}\ \exp\left[2\int_{H_o}^{H(u,v)} \{B(y)/\sigma^2_{HH}(y)\}dy\right]. \tag{8}$$

For the associated deterministic problem, i.e. $\eta_1(t) = \eta_2(t) = 0$, $\partial H/\partial t = 0$ gives constant energy loops such as limit cycles and separatrix, and the corresponding parameter values μ_o and μ_2 which satisfy $B(H) = 0$ can be obtained for values of $H \in (0, \nu_o^2/4)$ and $H \in (-\nu_o^2/4, 0)$ respectively for case (i) and case (ii). As before, the extrema of $W(H)$ given by $\partial\psi/\partial H = 0$ denote, so to speak, the continuation of the deterministic constant energy levels or the "limit cycles" and are given by

$$2\Lambda(H)B(H) + \sigma^2_{HH}(H)\frac{d\Lambda(H)}{dH} = 0 \tag{9}$$

The stability of such limit cycles is determined by the sign of $(\partial^2\psi/\partial^2 H)_{H = H_o}$. These results are shown in Fig. 2, where the left hand side of equation (9) is ploted against m and the most probable values are given by point m_2. In addition to these, the "fixed points" of the stochastic system are obtained by solving $\partial\psi/\partial u = 0$ and $\partial\psi/\partial v = 0$, and their stability is determined by the matrix of the second derivatives of ψ.

Exit Time Problems

In order to examine the stochastic stability of the equilibrium points from the Itô equation of the Hamiltonian H, it is important to determine the domain of attractions of the deterministic system. Knowing the domain of attraction we can say that the stochastic system losses stability (w.p.1) when the trajectories cross the boundary of this domain. In the sequel, the domain of attraction of the deterministic system is calculated for some regions in the parameter space for the case $P_2(\nu_o,u) = -(\nu_o^2/2)u + u^4/4$.

It is obvious when $\nu_o < 0$ and $\nu_2 < 0$, the trivial equilibrium point is asymptotically stable and when $\nu_o < 0$ and $\nu_2 = 0$, the deterministic system undergoes a Hopf bifurcation and stable limit cycle exist for $\nu_o < 0$ and $\nu_2 = 0$. Similarly, when $\nu_2 < 0$ and $\nu_o = 0$ the deterministic system

undergoes a simple bifurcation and two stable equilibrium points exist for $\nu_2 < 0$ and $\nu_o > 0$. Furthermore, applying Bendixson's criterion, i.e., if the divergence of the vector field in R^2 has a fixed sign (zero excluded) in a simply connected region D in R^2 the system has no closed orbits lying entirely in D, we have $(\nu_2 - u^2) < 0$ for $\nu_2 < 0$ implying no closed orbits encircling all three fixed points. The associated stochastic problems for Hopf and simple bifurcations were discussed by the author (1988a,1989). In this paper, attention is focussed on the region $\nu_o > 0$ and $\nu_2 > 0$.

Every periodic orbit for the deterministic system must encircle one or all three equilibrium points crossing the u-axis at (b,o). If $b \in (0,\sqrt{\nu_o})$ then the periodic orbit is a limit cycle encircling the equilibrium point $(\sqrt{\nu_o},0)$. Due to symmetry i.e., $P_2(u) = -P_2(-u)$, there exist another periodic orbit encircling the equilibrium point $(-\sqrt{\nu_o},0)$. Let the periodic orbit for the perturbed deterministic system be denoted as $\Gamma_\varepsilon(b_1,\nu_o,\nu_2)$. Along the solutions of Eq. (4) with $n_1 = n_2 = 0$, we have

$$\dot{H} = \varepsilon v^2 G(\nu_2,u) \tag{10}$$

and since $\Gamma_\varepsilon(b,\nu_o,\nu_2)$ is a closed path we also have

$$\int_{\Gamma_\varepsilon} \dot{H}dt = 0, \text{ i.e., } \tilde{F}(b,\varepsilon,\nu_o,\nu_2) = \int_{\Gamma_\varepsilon} v^2 G(\nu_2,u)\,dt = 0 \tag{11}$$

The function $\tilde{F}(b,o,\nu_o,\nu_2) \equiv \tilde{B}(b)$ may be written explicitly as

$$\tilde{B}(b) = \nu_2 \tilde{J}_o(b,\nu_o) - \tilde{J}_1(b,\nu_o) \tag{12}$$

where $\tilde{J}_o = \int_{\Gamma_o} v^2 dt$ and $\tilde{J}_1 = \int_{\Gamma_o} v^2 u^2 dt$

Thus, the solution of $\tilde{F}(b,o,\nu_o,\nu_2) \equiv \tilde{B}(b) = 0$ is given by

$$\nu_2 = \tilde{J}_1(b,\nu_o) \,/\, \tilde{J}_o(b,\nu_o) \tag{13}$$

Differentiating (12) yields

$$\frac{\partial \tilde{F}}{\partial \nu_2}(b,o,\nu_o,\nu_2) = \tilde{J}_o(b,\nu_o) \neq 0$$

which implies, by the implicit function theorem (IFT) that there exists a unique continuously differentiable function $\nu^*(b,\varepsilon,\nu_o)$ such that $\tilde{F}(b,\varepsilon,\nu_o,\nu^*(b,\varepsilon,\nu_o)) = 0$ for sufficiently small ε and

$$\nu^*(b,o,\nu_o) = \tilde{J}_1(b,\nu_o)/\tilde{J}_o(b,\nu_o) \tag{14}$$

Having shown the existence of limit cycle by IFT, we next proceed to list various periodic orbits present in the region $\nu_2 > 0$ and $\nu_o > 0$. However, it is convenient to employ in place of b another parameter H, which corresponds to the energy level $H = -(b^2/2)\,(\nu_o - b^2/2)$. This change of parameters is justified, since $\partial H/\partial b = -\,b(\nu_o - b^2)$ is zero only at $b = \pm\sqrt{\nu_o}$ and 0, which are the fixed points. Thus, for $H \in [-\nu_o^2/4, 0]$, making $B(H) \equiv 0$ yields

$$\nu_2(H,\nu_o) = J_1(H)/J_o(H) = R(H(m)) \cdot \nu_o \tag{15}$$

where $$J_o(H) = \int_{u_o^-(H)}^{u^+(H)} \left(2H + \nu_o u - \frac{u^4}{2}\right)^{1/2} du$$

$$= (2/3)[\nu_o/(2-m)]^{3/2}[(2-m)E(m) - 2(1-m)F(m)]$$

$$J_1(H) = \int_{v_o^-(H)}^{u^+(H)} \left(2H + \nu_o u - \frac{u^4}{2}\right)^{1/2} u^2 du$$

$$= (4/15)\,[\nu_o/(2-m)]^{5/2}[2(m^2 - m + 1)E(m) + (2-m)(m-1)F(m)]$$

and $$m = 2(\nu_o^2 + 4H)^{1/2}/[\nu_o + (\nu_o^2 + 4H)^{1/2}]\ ,\quad m \in [0,1]$$

The following limits

$$\lim_{m \to 0} \nu_2(H(m),\nu_o) = \nu_o \text{ and } \lim_{m \to 1} \nu_2(H(m),\,\nu_o) = (4/5)\nu_o \tag{16}$$

agree with the calculations of Hopf bifurcation (local analysis) and saddle-loop (using Melnikov integrals) respectively in the region $\nu_o > 0$ and $\nu_2 > 0$ (see, for example, Guckenheimer and Holmes 1983). Furthermore, it can be shown as in Carr (1981) that the limit cycle encircling each of the nontrivial fixed points is unique for $H \in [-\nu_o^2/4, 0]$.

It should be further noted that as the noise terms tend to zero, the extrema given by equation (9) tend to the steady state solution of the deterministic system given by (15). This further emphasizes the fact that the most probable values are so to speak the continuation of the deterministic steady states. The critical values of both equations (9) and (15) are given in Fig. 2. However, in this paper the domain of attraction is obtained using the critical points of the deterministic equation (15) which are shown in Fig. 3. It can be seen that the noise terms lower the critical values.

Now that we have established the domain of attractions, we formulate the exit time problem associated with the nontrivial fixed points in the region $\nu_2 > 0$, $\nu_o > 0$ and determine the probabilistic information concerning the time T when the stochastic response process first passes out of a local domain of attraction of the fixed points. Since there are two simply connected domains for $H \in [-\nu_o^2/4, 0]$ we shall concentrate on the region $\nu_2 \in [\nu_o, (4/5)\nu_o]$. Furthermore, for a specific value of ν_2 and ν_o there is a unique value of H or equivalently $m \in [0,1]$ given by $m = R^{-1}(\nu_2/\nu_o)$ which defines the boundary Γ of the domain (see Fig. 3).

$$dm = \mu_m(m)\, dt + \sigma_{mm}(m) dw \tag{17}$$

where $$\mu_m(m) = \left(\frac{1}{\Gamma(m)}\right)\left[B(m) + \frac{1}{2}\frac{d}{dm}\left[\sigma^2(m)\right]\right], \quad \Gamma(m) = \frac{m\nu_o^{3/2}F(m)}{(2-m)^{5/2}}$$

$$\sigma_{mm}^2(m) = \left(\frac{1}{\Gamma(m)}\right)\left[\sigma^2(m)\right], \quad \sigma^2(m) = \sigma_{HH}^2 \frac{(2-m)^3}{m\nu_o^2}, \quad 0 \le m \le 1$$

Suppose that at time $t = 0$, the state of the system corresponds to some point defined by m_o within D. When the random disturbance is applied, we are interested in the time T it takes for a trajectory at m_o to reach the boundary Γ of D for the first time, i.e.

$$T = \min \{t: m(t) \in \Gamma \mid m(0) = m_o\}, \; m_o \varepsilon D \tag{18}$$

Now defining the probability that a trajectory has not reached the boundary Γ during time interval τ as

$$P(\tau, m_o) = P_r\{\tau < T(m_o)\} \tag{19}$$

the Kolmogorov's (backward) equation is written as

$$\frac{\partial P}{\partial \tau}(\tau, m_o) = \mu_m(m_o) \frac{\partial P}{\partial m_o} + \frac{1}{2} \sigma^2_{mm} (m_o) \frac{\partial^2 P}{\partial m_o} = L[P(\tau, m_o)] \tag{20}$$

with the initial and boundary conditions

$$P(o, m_o) = 1 \qquad m_o \in D \quad , \quad P(\tau, m_c) = 0 \qquad m_c \in \Gamma$$

The distribution function of the first passage time $P_r[\tau = T] = 1 - P(\tau, m_o)$ and the corresponding Pontriagin equation for the nth moment is given by

$$L[M_n(m_o)] = - nM_{n-1}(m_o) \text{ and } M_n(m_c) = 0 \tag{21}$$

Finally, the mean first passage time can be written as the solution of

$$L[M_1(m_o)] = \frac{1}{2} \sigma^2_{mm} (m_o) \frac{\partial^2 M_1}{\partial m_o{}^2}(m_o) + \mu_m(m_o) \frac{\partial M_1}{\partial m_o}(m_o) = -1 \tag{22}$$

with $M_1(m_c) = 0$.

In addition to the boundary condition $M_1(m_c) = 0$, a boundedness condition at $m_o = 0$, $M_1(0) < \infty$, is usually imposed on the solution to uniquely determine $M_1(m_o)$. This condition by itself implies that the left boundary $m_o = 0$ is not an absorbing boundary. This condition may be violated if the noise term $\sigma^2_{mm}(m_o)$ vanishes at $m_o = 0$, and cannot then be used to obtain the solution. It is, therefore, important to understand the behavior of the diffusion process m(t) near the boundary $m_o = 0$ according to various Feller classification (see, for example, Feller, 1954; Itô and McKean, 1964; Karlin

and Taylor, 1981). To this end, consider an interval $(0, m_c]$ and let $\Delta \in (0, m_c)$ be an interior point such that m(t) is a regular diffusion process in $[\Delta, m_c]$. Putting T_x as the hitting time of x, we can define $\bar{M}_1(m_o)$ as the mean time to reach either Δ or m_c, i.e.

$$\bar{M}_1(m_o) = E\{T_\Delta \wedge T_{m_c} | m(0) = m_o\} \text{ for } \Delta < m_o < m_c$$

and can be evaluated as

$$\bar{M}_1(m_o) = 2[1 - \alpha(m_o, m_c)/\alpha(\Delta, m_c)] \int_{m_o}^{m_c} \Gamma(\eta)\alpha(\eta, m_c) \exp[\Phi(\eta)]d\eta$$

$$+ 2\alpha(m_o, m_c)/\alpha(\Delta, m_c) \int_{\Delta}^{m_o} \Gamma(\eta)\alpha(\Delta, \eta) \exp[\Phi(\eta)]d\eta \qquad (23)$$

where

$$\Phi(\eta) = \int_0^\eta \frac{2B(x)}{\sigma^2(x)} dx \ , \quad \alpha(z_1, z_2) = \int_{z_1}^{z_2} \exp[-\Phi(x)] \frac{dx}{\sigma^2(x)}$$

In order to classify the boundary behavior, consider the following quantities γ_o and β_o, which can be respectively defined as roughly the measure of time to reach the left boundary 0 starting from an interior point $m_o \in (0, m_c)$ and the measure of time to reach an interior point m_o starting from the left boundary 0. They are defined as

$$\gamma_o = \int_0^z \frac{1}{\sigma^2(x)} \{\int_x^z \Gamma(\eta) \exp[\Phi(\eta)]d\eta\} \exp[-\Phi(x)]dx \qquad (24)$$

$$\beta_o = \int_0^z \Gamma(\eta) \{\int_\eta^z \exp[-\Phi(x)] \frac{dx}{\sigma^2(x)}\} \exp[\Phi(\eta)]d\eta \qquad (25)$$

where z is an interior point. The Feller classification of the left boundary, 0, in terms of γ_o and β_o is as follows:

(1) The left boundary 0 is <u>regular</u>, if $\gamma_o < \infty$ and $\beta_o < \infty$. The process can both enter and leave from the boundary, 0. In other words, the process

starting from an interior point can reach the boundary with some positive probability in finite time. Similarly the process starting from the boundary can reach an interior point with some positive probability infinite time.

(2) The left boundary 0 is an <u>exit</u>, if $\gamma_o < \infty$ and $\beta_o = \infty$. The process starting from an interior point can reach the boundary with some positive probability, but starting at 0, it is impossible to reach any interior point $m_o \in (0, m_c)$. Furthermore, the exit boundary is either a trap or an absorbing point.

(3) The left boundary 0 is an <u>entrance</u>, if $\gamma_o = \infty$ and $\beta_o < \infty$. An entrance boundary cannot be reached from any interior point, i.e. the probability is zero that the process starting at an interior point can reach an entrance boundary. Furthermore, the process starting from an entrance boundary moves at once to the interior never to return to it.

(4) The left boundary 0 is <u>natural</u>, if $\gamma_o = \infty$ and $\beta_o = \infty$. The process starting from an interior point cannot reach the boundary in finite time and the process cannot reach any interior point starting from the natural boundary.

Using this classification, Kozin and Sunahara (1987) established some stability properties of the singular point, 0. In the similar lines we establish an instability condition for the singular point 0.

<u>Theorem</u>:

Let the singular point 0 be the left boundary of the interval $(0, m_c]$. The singular point is unstable in probability, i.e. the event $\{\sup_{t>0} |m(t:m_o, 0)| < \varepsilon\}$ has probability zero for all $m_o > 0$, if the following conditions are satisfied

i) 0 is an entrance boundary, i.e. $\gamma_o = \infty$ and $\beta_o < \infty$

ii) m_c is a regular or an exit boundary, i.e. $\gamma_{m_c} < \infty$

where

$$\gamma_{m_c} = \int_z^{m_c} \frac{1}{\sigma^2(x)} \left\{\int_z^x \Gamma(n) \exp[\Phi(n)]dn\right\} \exp[-\Phi(x)]\, dx . \tag{26}$$

Moreover, the mean time to reach m_c is given by

$$M_1(m_o) = 2 \int_{m_o}^{m_c} \Gamma(\eta) \left\{\int_{\eta}^{m_c} \exp[-\Phi(x)] \frac{dx}{\sigma^2(x)}\right\} \exp[\Phi(\eta)]d\eta \tag{27}$$

and $M_1(0) < \infty$.

Proof:

The proof follows the boundary behavior according to the above classification. From the definition of β_o it is obvious that $\beta_o < \infty$ implies

$$\lim_{\Delta \to 0} \int_{\Delta}^{z} \Gamma(\eta) \exp[\Phi(\eta)]\, d\eta < \infty \ , \ \Delta < z < m_c$$

and hence for an entrance boundary the following relationship

$$\gamma_o + \beta_o = \lim_{\Delta \to 0} \left\{\alpha(\Delta,z) \int_{\Delta}^{z} \Gamma(\eta) \exp[\Phi(\eta)]\, d\eta\right\}$$

yields $\lim_{\Delta \to 0} \alpha(\Delta,z) = \infty$. Thus for some m_o, $0 < m_o < z < m_c$, the Pr $\{T_z < T_{o+} \mid m(0) = m_o\}$ is given by

$$\lim_{\Delta \to 0} [1 - \alpha(m_o,z)/\alpha(\Delta,z)] \to 1$$

Moreover, since $\beta_o < \infty$ we conclude that, the process starting at any $m_o \in [o,z]$ reaches z in finite time with probability one. If the right boundary is regular, the z is replaced by m_c. The right boundary is attractive and attainable when m_c is an exit point. Thus for both these cases

$$\Pr\left\{\sup_{t>0} |m(t:m_o,0)| < \varepsilon\right\} = 0, \quad \text{for all } m_o > 0$$

Now using the fact $\lim_{\Delta \to 0} \alpha(\Delta,m_c) = \infty$ the exit time can be obtained from equation (23) as (27).

For the problem under consideration, numerical calculations indicate that $\gamma_o = \infty$, $\beta_o < \infty$ and $\gamma_{m_c} < \infty$ and thus the singular point 0 is unstable in

probability. The mean exit time to reach m_c can be calculated using equation (27). These results are shown in Fig. 4.

Conclusions

In this paper, a special case of co-dimension two stochastic bifurcation associated with nonsemi-simple double zero eigenvalues is examined. The normal form associated with this case is two dimensional and corresponds to the stochastically perturbed van der Pol - Duffing equation. The stochastic averaging method appropriate for the Itô equations is applied to obtain a one dimensional Itô equation for the Hamiltonian. The probability density, and its extrema are obtained. It is found that the extrema of the density function correspond to the least probable value from which the process leaves rather quickly. The critical extrema values m_c of the stochastic system approach the values of m at which the deterministic unstable limit cycle exist as the noise terms tend to zero. Furthermore, the effect of additive and multiplicative noise terms on the values of m_c are demonstrated and it is shown that the noise term lower these critical values. In order to examine the stochastic stability of the equilibrium points from the Itô equation for Hamiltonian, the domain of attraction of the deterministic system is obtained. The stability in probability and the mean exit time are calculated using the domain of attraction. The stability in probability is obtained by examining the boundary behavior of the one dimensional diffusion process. An instability theorem is stated and proven. It is found that the nontrivial fixed points are unstable in probability. The mean exit time to reach the boundaries of the domain of attraction is determined for various noise levels.

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Figure Captions

Fig. 1a The stationary density vs m for parametrically perturbed system.

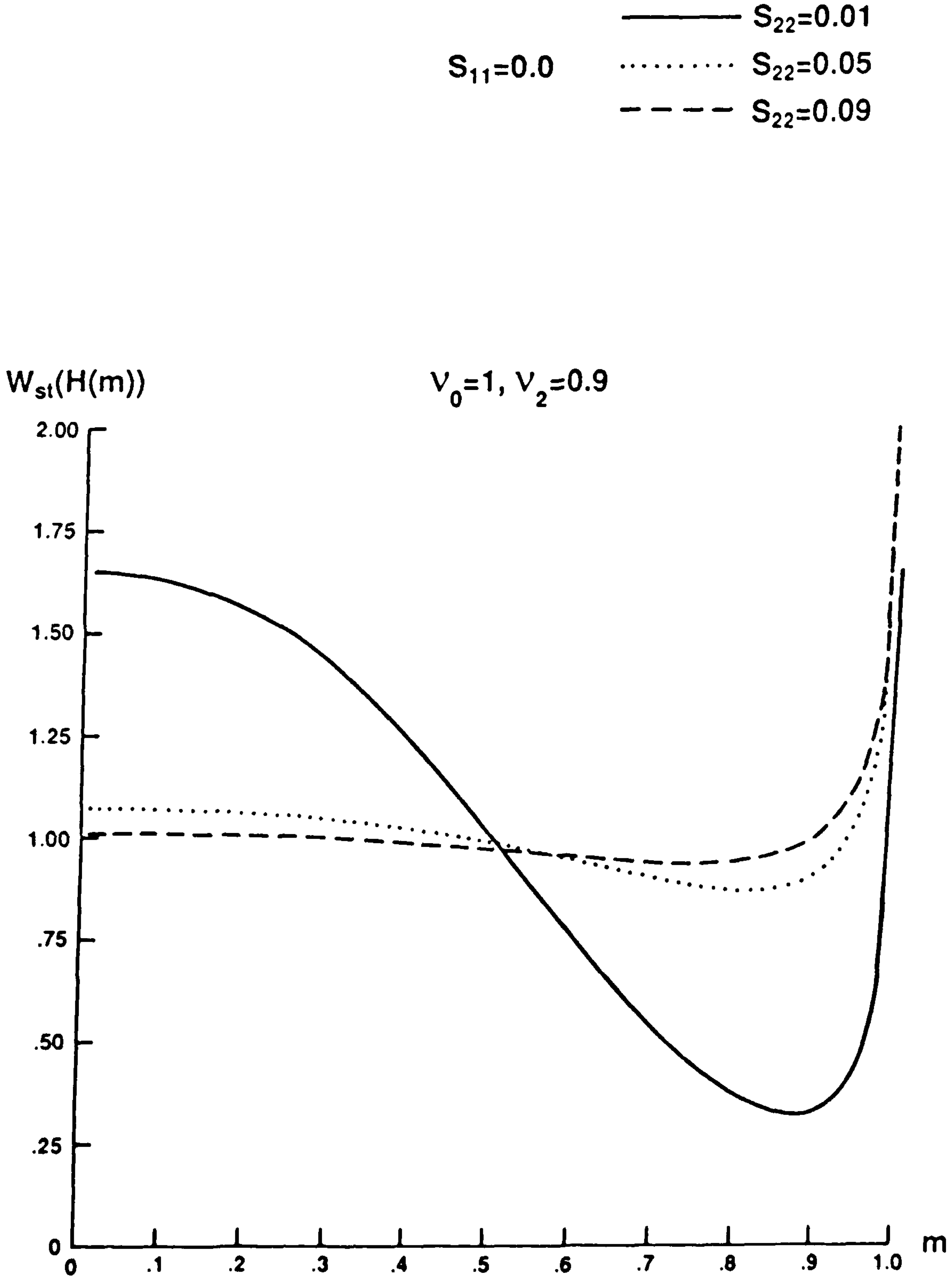
Fig. 1b The stationary density vs m for system with both parametric and external excitations.

Fig. 2a The comparison of critical points of the stochastic system with that of the deterministic system.

Fig. 2b The critical points vs ν_2 for the stochastic case.

Fig. 3 The right boundary of the domain of attraction vs ν_2.

Fig. 4 The mean exit time vs m_0.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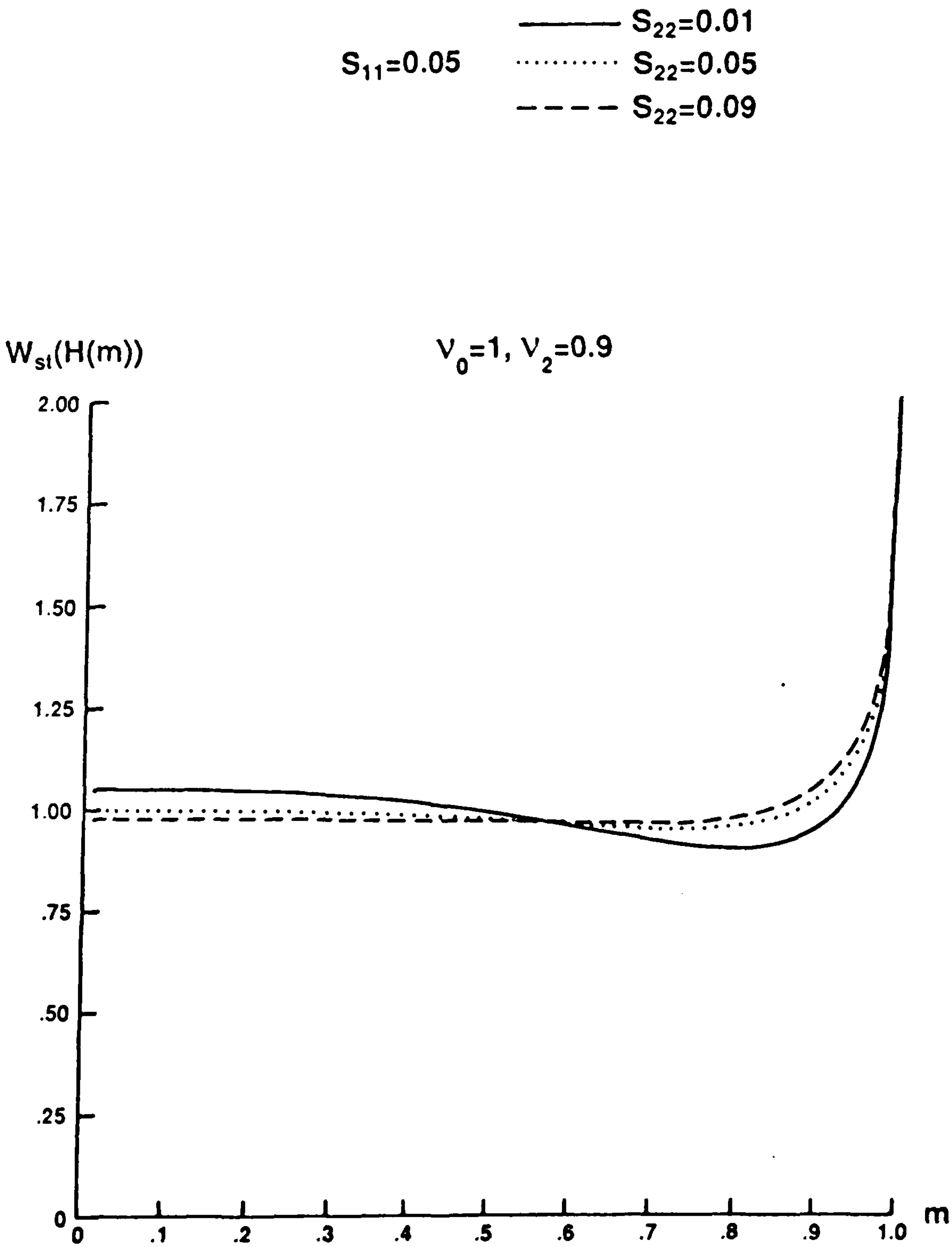
S22=0.05
S22=0.09
S11=0.0
Wst(H(m))
V0=1, V2=0.9
2.00
1.75
1.50
1.25
1.00
.75
.50
.25
0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
m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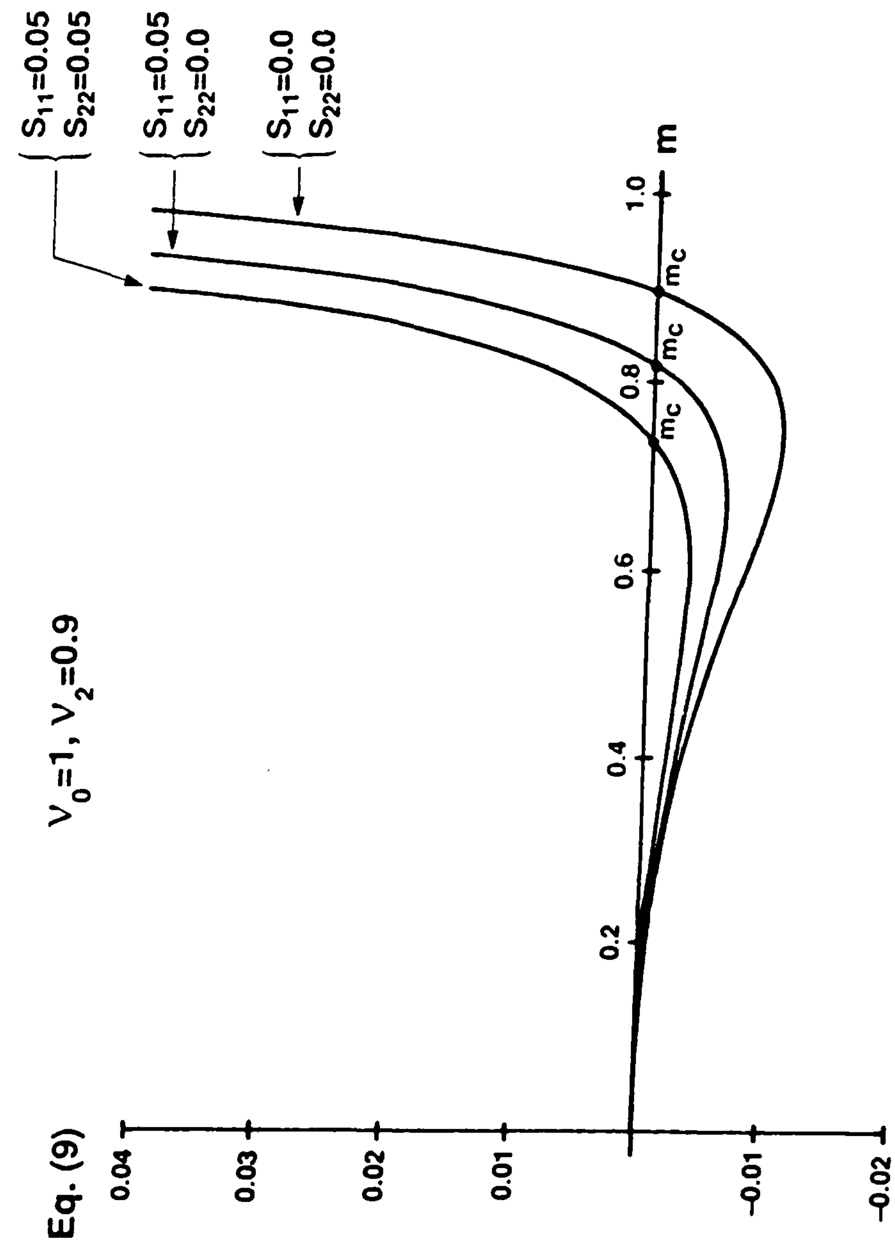
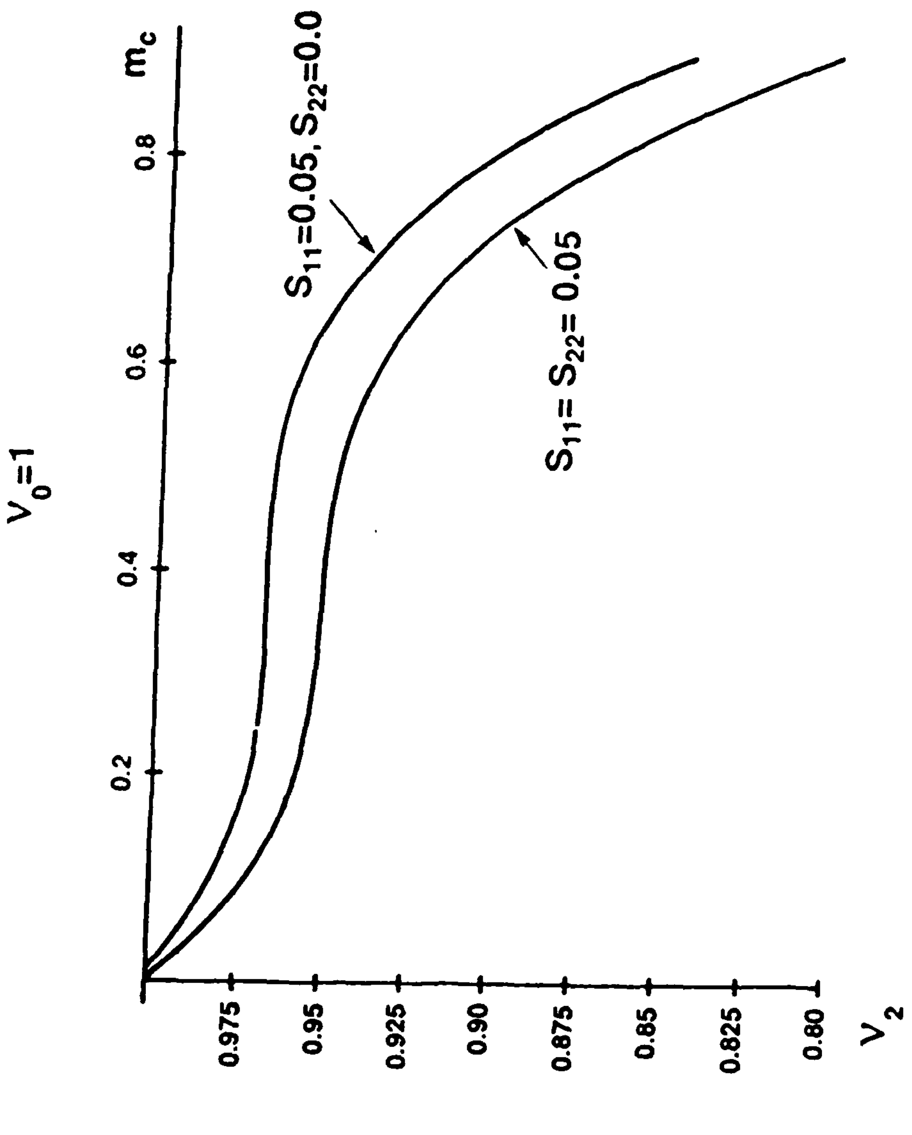
S_{22}=0.01
S_{22}=0.05
S_{22}=0.09
$W_{st}(H(m))$
V_0=1, V_2=0.9
2.00
1.75
1.50
1.25
1.00
.75
.50
.25
0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
m


Eq. (9)
0.04
0.03
0.02
0.01
−0.01
−0.02
0.2
0.4
0.6
0.8
1.0
m
m_c
m_c
m_c
V_0=1, V_2=0.9
S_{11}=0.05 S_{22}=0.05
S_{11}=0.05 S_{22}=0.0
S_{11}=0.0 S_{22}=0.0


$V_0=1$
m_c
0.2
0.4
0.6
0.8
V_2
0.975
0.95
0.925
0.90
0.875
0.85
0.825
0.80
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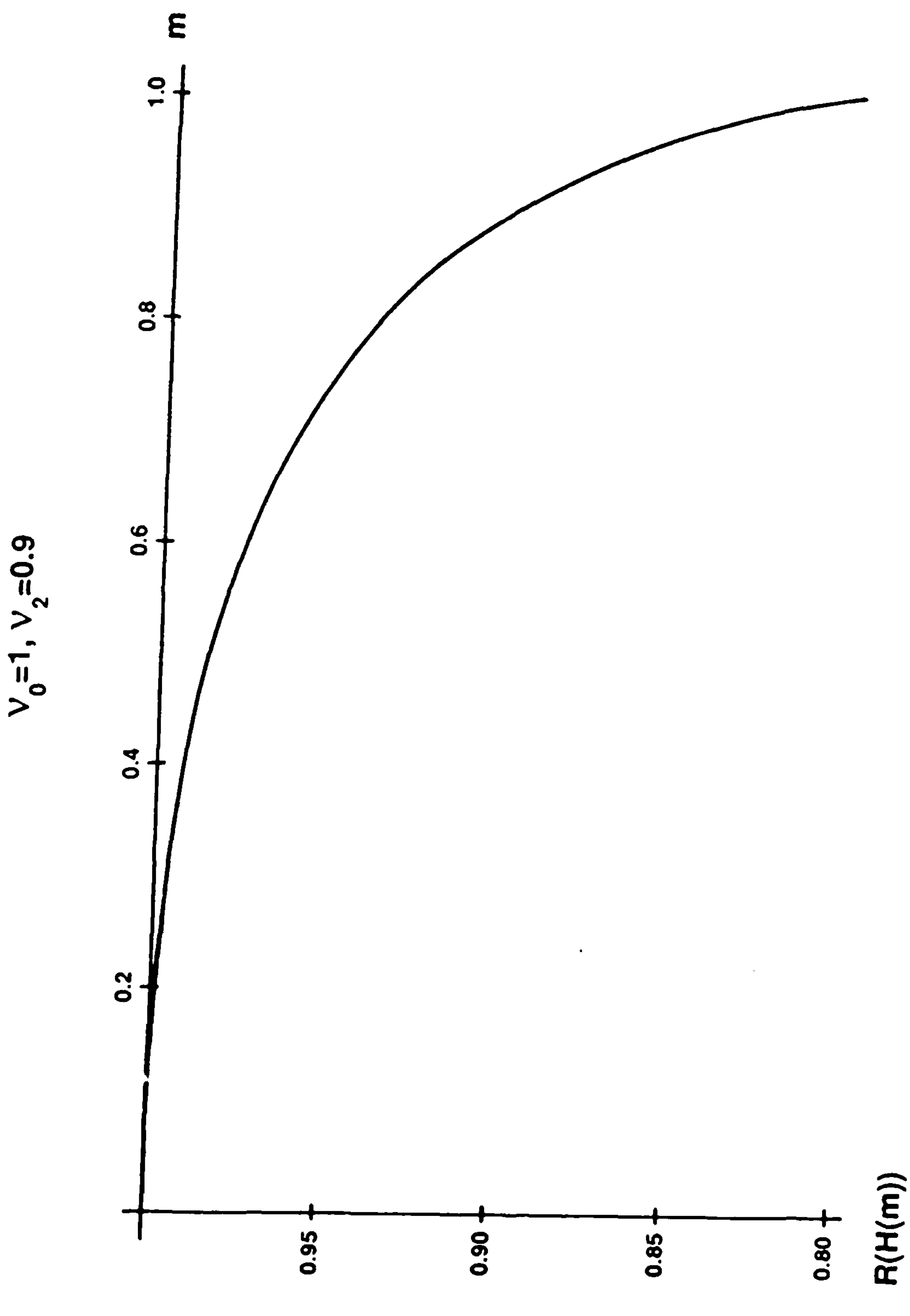
$S_{11}=S_{22}=0.05$


V_0=1, V_2=0.9
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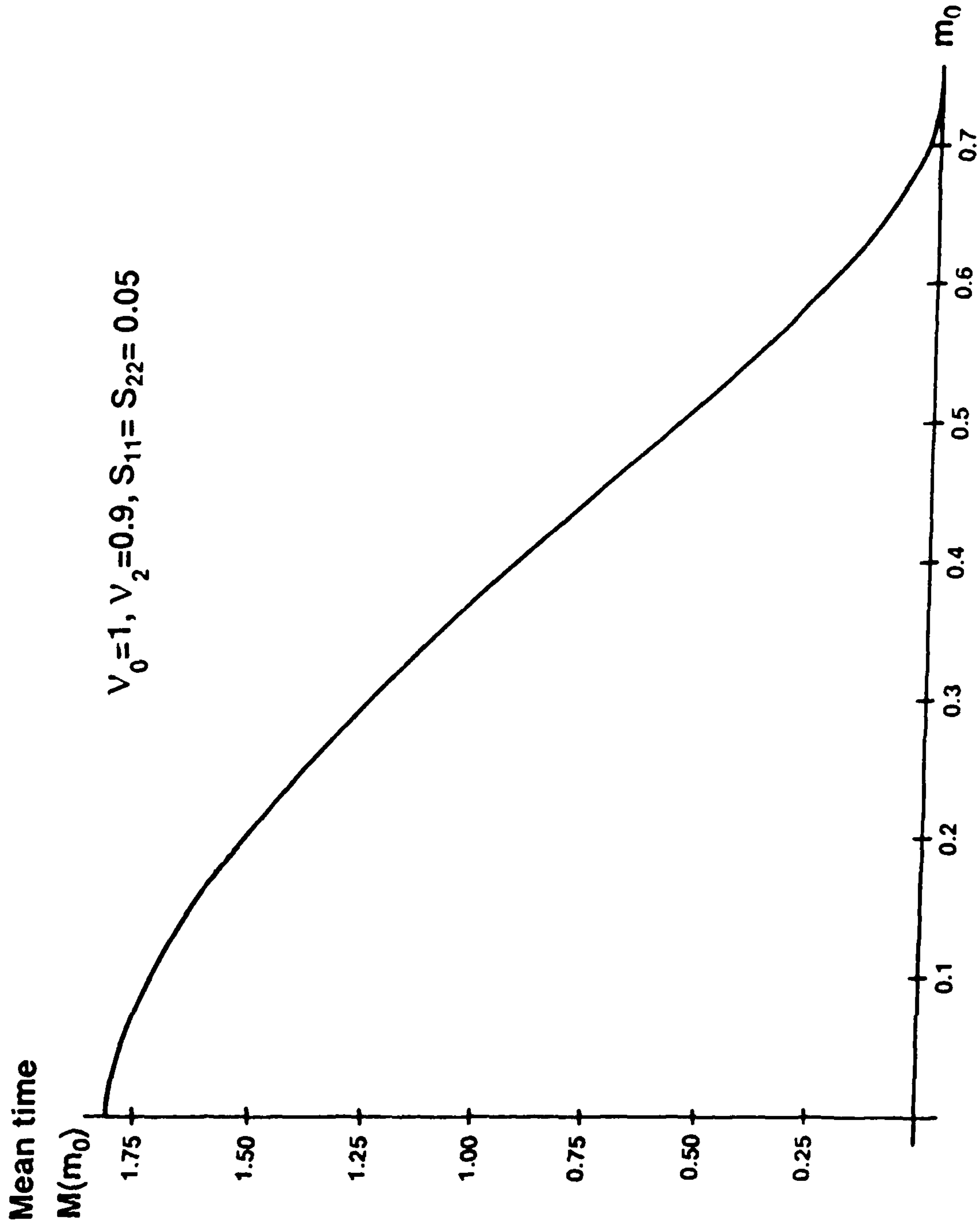
0.8
0.6
0.4
0.2
R(H(m))
0.95
0.90
0.85
0.80


V_0=1, V_2=0.9, S_{11}= S_{22}= 0.05
Mean time
$M(m_0)$
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.1
0.2
0.3
0.4
0.5
0.6
0.7
m_0

APPENDIX C

I. Introduction

Recently Hui and Tobak[1] analyzed the Hopf bifurcation that results when the steady flight of an aircraft becomes unstable in pitch by increasing the angle of attack. For the case of a double wedge aerofoil it was found that, in addition to Hopf bifurcation, degenerate Hopf bifurcation can also take place due to the violation of a certain transversality condition[2]. Since such degenerate bifurcation is nongeneric, Sri Namachchivaya and Van Roessel[3] made use of the results of singularity theory to unfold these bifurcations.

The purpose of this note is to extend these results on the nonlinear analysis of a pitching aircraft at high angles of attack. It will be shown that, in addition to the above mentioned co-dimension one and two bifurcations, there exist co-dimension two bifurcations associated with a double zero eigenvalue. Usually a zero eigenvalue implies a simple bifurcation. In the case of a double zero eigenvalue with non-semisimple Jordan form and no further degeneracy, two parameters are required for a complete universal unfolding[4]. All possible bifurcations that take place in the neighborhood of this bifurcation point will be obtained by making use of these unfolding parameters. A family of limit cycles may branch off from the equilibrium surface in the vicinity of such a critical point. For the equations of motion for a pitching wedge such a double zero eigenvalue does occur at certain critical values of the system parameters. The partial unfolding for this case is carried out below.

Consider an aircraft in steady flight at an angle of attack σ. Suppose some disturbances take place at time $t = 0$, e.g. due to a change in the flap deflection angle; the aircraft will subsequently undergo an unsteady motion relative to its steady flight. Such an unsteady motion of the aircraft modifies the air flow and hence the aerodynamic forces on the aircraft which

in turn determine its motion. Thus the aircraft's subsequent motion can only be determined by simultaneously solving the unsteady flow equations of the air and the equations of motion of the vehicle as a rigid body, aeroelastic effects being assumed negligible.

Although simultaneously solving the coupled equations in principle represents an exact approach to the problem of arbitrary maneuvers, it is inevitably a very difficult and costly approach. In classical aerodynamics, the traditional approximate approach is to assume the pitching motion to be a small amplitude periodic oscillation consisting of simple harmonics. On this basis the flow equations are decoupled from the inertia equation, and are linearized to determine the aerodynamic response to such an harmonic motion. The so-called aerodynamic coefficients thus obtained are then used to predict the motion of the aircraft. Even though this approach ignores the time-history effects on the flow field and the aircraft motion, it gives a good approximation for calculating the aerodynamic response from the unsteady flow equations and hence the pitching moment. This approximation which has been adopted by Hui and Tobak[1] and Sri Namachchivaya and Van Roessel[3] in their investigations of this problem is used in this paper.

II. Statement of the Problem

Consider an aircraft in flight, free to undergo a single degree of freedom pitching motion. The equations of pitching motion can be expressed as

$$\frac{d\alpha}{dt} = \dot{\alpha}, \qquad I\frac{d\dot{\alpha}}{dt} = M(t), \tag{1}$$

where α is the instantaneous angle of attack, I is the moment of inertia of the vehicle about the pivot axis, and $M(t)$ is the pitching moment at instantaneous time t of the aerodynamic forces about the same axis. When the

motion is slowing varying[5], the pitching moment $M(t)$ may be characterized with sufficient accuracy by the instantaneous angle of attack $\alpha(t)$ and the instantaneous rate of change of the angle of attack $\dot{\alpha}(t)$. Suppose $\alpha = \sigma$ is an equilibrium state of the system of Eqs. (1); then, putting $\alpha(t) = \sigma + \psi(t)$, the variational equations about the equilibrium position can be written as

$$\frac{d\psi}{dt} = \dot{\psi} \quad , \quad \frac{d\dot{\psi}}{dt} = M(t) \tag{2}$$

where ψ is the angular displacement of motion measured from the angle of attack σ of the steady flight. It is assumed that the moment required to trim the aircraft at σ has been accounted for, so that $M(t)$ is a measure of the perturbation moment only and is determined from the instantaneous surface pressure. As noted earlier following the mathematical modeling approach of Tobak and Schiff[5], instantaneous pitching moment can be given as

$$M(t) = \frac{\rho_\infty V_\infty^{\;2}}{2} \bar{S}L \left[C_m(0,0,\sigma,h) - C_m(\psi,\dot{\psi},\sigma,h)\right]$$

where ρ_∞ and V_∞ are the free stream density and velocity, respectively; $\bar{S}$ and L are the reference area and length; and h represents the distance between the apex and the pivot position as defined in Fig. 1. The function $C_m(\psi,\dot{\psi},\sigma,h)$ represents the pitching moment coefficient of the aerodynamic forces about the pivot axis and $C_m(0,0,\sigma,h)$ is its steady value at a fixed angle of attack σ. Even though C_m depends on the flight Mach number M_∞ , the specific heats of the air and the aircraft shape, these parameters will be considered as "passive" parameters in this analysis. For a <u>finite amplitude, slow, pitching motion</u> with angular displacement $\psi(t)$ around a mean angle of attack σ, with terms of $O(\dot{\psi}^2,\ddot{\psi})$ assumed negligible, we can write[1]

$$-C_m(\psi,\dot{\psi},\sigma,h) = f(\sigma + \psi,h) + g(\sigma + \xi,h)\dot{\psi},$$

which reduces the second of Eqs. (2) to

$$\frac{d\dot{\psi}}{dt} = F(\psi,\dot{\psi},\sigma,h)$$

where

$$F(\psi,\dot{\psi},\sigma,h) = \frac{M(t)}{I} = \kappa\left[f(\sigma + \psi,h) - f(\sigma,h) + g(\sigma + \psi,h)\dot{\psi}\right] ,$$

$$\kappa = \frac{I}{2I} \rho_\infty V_\infty^2 \bar{S}L$$

Equations (2) represent a pair of autonomous differential equations in R^2 the trivial solution of which is $\psi = 0$. The objective of this investigation is to understand the stability and the bifurcation behavior of the stationary solutions of Eqs. (2) as the system parameters σ and h are varied.

III. Bifurcation of Fixed Points

The functions $f(\sigma,h)$ and $g(\sigma,h)$ are related to the stiffness derivative $S(\sigma,h)$ and the damping derivative $D(\sigma,h)$ of classical aerodynamics as follows:

$$\kappa \frac{\partial f}{\partial \sigma} (\sigma,h) = - S(\sigma,h) \ , \ \kappa g(\sigma,h) = - D(\sigma,h)$$

Introducing new state variables $\bar{x} = \psi$, $\bar{y} = \dot{\psi}$, Eqs. (2) may be written in the form

$$\bar{x}' = \bar{y},$$

$$\bar{y}' = \bar{\mu}_1\bar{x} + \bar{\mu}_2\bar{y} + \bar{p}_0\bar{x}^2 + \bar{q}_0\bar{x}\ \bar{y} + \bar{p}_1\ \bar{x}^3 + \bar{q}_1\bar{x}^2\bar{y}, \tag{3}$$

where $\bar{\mu}_1 = - S(\sigma,h)$, $\bar{\mu}_2 = - D(\sigma,h)$, $\bar{p}_0 = - \frac{1}{2} \frac{\partial S}{\partial \sigma} (\sigma,h)$,

$$\bar{q}_0 = -\frac{\partial D}{\partial \sigma}(\sigma,h)\ ,\ \bar{p}_1 = -\frac{1}{3!}\frac{\partial^2 S}{\partial \sigma^2}(\sigma,h)\ ,\ \bar{q}_1 = -\frac{1}{2}\frac{\partial^2 D}{\partial \sigma^2}(\sigma,h).$$

It is evident that the nongeneric case $\bar{\mu}_1 = \bar{\mu}_2 = 0$ i.e., $D(\sigma_c,h_c) = 0$ and $S(\sigma_c,h_c) = 0$, gives rise to a double zero eigenvalue with non semisimple Jordan form as the Jacobian. It is well known that the damping and the stiffness derivatives are respectively quadratic and linear in h, i.e.,

$$D(\sigma,h) = D_0(\sigma) + D_1(\sigma)\,h + D_2(\sigma)h^2$$

$$S(\sigma,h) = S_0(\sigma) + S_1(\sigma)h$$

Furthermore, the qualitative variations of the quantities $D(\sigma,h)$ and $S(\sigma,h)$ with σ and h can be found in Hui[6] for double-wedge aerofoil. The variations of the components of S and D, namely S_0, S_1, D_0, D_1, and D_2 are given graphically in Sri Namachchivaya and Van Roessel[3]. The critical values of σ and h are obtained by letting $D = S = 0$ in the above expressions. The critical parameter values and the various coefficients needed for the analysis are given in Table 1 for $\tau_0 = \tau_1 = 5°$. Introducing new variables x, y and new time t,

$$x = \left(\frac{\bar{q}_0^{\,2}}{\bar{p}_0^{\,2}}\right)\bar{x}\ ,\ y = \left(\frac{\bar{q}_0^{\,3}}{\bar{p}_0^{\,2}}\right)\bar{y}\ ,\quad t = \left(\frac{p_0}{q_0}\right)\bar{t}\ ,\quad p_0,q_0 \neq 0$$

yields

$$\dot{x} = y, \tag{4}$$
$$\dot{y} = \mu_1 x + \mu_2 y + x^2 + xy + p_1 x^3 + q_1 x^2 y,$$

where $\mu_1 = \left(\frac{\bar{q}_0}{\bar{p}_0}\right)\bar{\mu}_1\ ,\ \ \mu_2 = \left(\frac{\bar{q}_0}{\bar{p}_0}\right)\bar{\mu}_2\ ,\ \ p_1 = \frac{\bar{p}_1}{\bar{q}_0^{\,2}}$ and $q_1 = \frac{\bar{q}_1 p_0}{\bar{q}_0^{\,3}}$.

The theory of normal forms deals with finding near identity coordinate

transformations, which simplify the analytic expressions of the nonlinear terms. The resulting simplified nonlinear equations are said to be in normal form. Eqs. (4) are in normal form since the expression for the normal form, for a nonlinear system with quadratic and cubic nonlinearities and a double-zero non-semisimple Jordan block, is identical to that of Eqs. (4). Furthermore, when the quadratic nonlinearities are not identically zero, the higher-order terms (i.e., cubic terms) do not contribute to qualitative changes in the phase portrait. Thus, a simplified set of equations

$$\dot{x} = y\,, \qquad (5)$$

$$\dot{y} = \mu_1 x + \mu_2 y + x^2 + xy\,,$$

which reveals all the principal phenomena contained in the general problem will be analyzed. In Eqs. (5), μ_1 and μ_2 are the unfolding parameters and are related to the determinant and the trace respectively of the linear operator of Eq. (3). We first seek the fixed points of Eqs. (5) which are given by $(x_o,y_o) = (0,0)$ and $(x_o,y_o) = (-\mu_1,0)$. Putting $x = x_o + u$ and $y = y_o + v$, the variational equations about the fixed point can be written as

$$\dot{u} = v, \qquad (6)$$

$$\dot{v} = \alpha_1 u + \alpha_2 v + u^2 + uv,$$

where $\alpha_1 = \mu_1 + 2x_o$, and $\alpha_2 = \mu_2 + x_o$. The eigenvalues of the fixed point are given by

$$\lambda_{1,2} = \frac{\alpha_2}{2} \pm \sqrt{\alpha}\quad,\quad \alpha = \frac{\alpha_2^2}{4} + \alpha_1. \qquad (7)$$

It is evident from Eq. (7) that the fixed point is asymptotically stable if $\alpha_1 < 0$ and $\alpha_2 < 0$, and goes through a Hopf bifurcation at $\alpha_2 = 0$ and $\alpha_1 < 0$. Thus, making use of the transformation

$$\binom{u}{v} = \begin{pmatrix} 0 & 1 \\ \sqrt{-\alpha_1} & 0 \end{pmatrix} \binom{\xi}{\eta},$$

Eq. (6) in the neighborhood of $\alpha_2 = 0$ can be written as

$$\dot{\xi} = \alpha_2 \xi - \sqrt{-\alpha_1}\, \eta + \frac{1}{\sqrt{-\alpha_1}} \eta^2 + \eta\, \xi, \qquad (8)$$
$$\dot{\eta} = \sqrt{-\alpha}\, \xi.$$

Now the formulas for Hopf bifurcation given by Guckenheimer and Holmes[7] are used to obtain the equation governing the bifurcating path as

$$2Ra^3 + \alpha_2 a = 0 \quad , \quad R = -\frac{1}{8\alpha_1} ,$$

where "a" represents the amplitude of the bifurcating periodic solution. Since $\alpha_1 < 0$ it is obvious that the fixed points undergo an unstable subcritical Hopf bifurcation when $\alpha_2 = 0$. Moreover the fixed point goes through a simple transcritical bifurcation at $\alpha_1 = 0$ and $\alpha_2 = 0$. It may be noted that the fixed point, a stable node for $\alpha_2 < 0$ and an unstable node or $\alpha_2 > 0$, becomes a saddle-node at $\alpha_1 = 0$ while undergoing a transcritical bifurcation.

IV. Global Bifurcations

It is clear from the analysis performed thus far and Fig. 2, that the phase portraits in region (3) and (4) are not homeomorphic since the former has a limit cycle and the latter does not. For similar reasons, regions (8) and (7) are also not homeomorphic. Hence, there must be additional global bifurcations occurring in which the nature of the fixed points do not change, but the phase portraits as a whole undergoes a topological change. In this section such global bifurcations are examined. The fixed point is a saddle point when $\alpha_1 > 0$, and making use of the transformation

$$u = \epsilon^2 z_1 \ , \quad v = \epsilon^3 z_2 \ , \quad \alpha_1 = \epsilon^2 \nu_1 \ , \quad \alpha_2 = \epsilon^2 \nu_2 \text{ and } \tau = \epsilon t.$$

yields

$$\frac{dz_1}{d\tau} = z_2$$

$$\frac{dz_2}{d\tau} = \nu_1 z_1 + z_1{}^2 + \varepsilon\,(\nu_2 z_2 + z_1 z_2) \tag{10}$$

where $|\varepsilon| << 1$ and $\varepsilon \leq 0$. For $\varepsilon \to 0$ the above equations become an integrable Hamiltonian system with Hamiltonian

$$H(z_1, z_2) = \frac{z_2^2}{2} - \nu_1 \frac{z_1{}^2}{2} - \frac{z_1{}^3}{3}$$

and $(z_1 = z_2 = 0)$ is a stable point and possesses a "saddle connection". Since the Hamiltonian is conserved the level curves H = constant are solutions of Eqs. (10) with $\varepsilon = 0$. Furthermore, the value of the Hamiltonian at the saddle point is $H(0,0) = 0$, and the points of intersections of the saddle loop with the axis $z_2 = 0$ are $z_1 = 0$ and $z_1 = -\frac{3}{2}\nu_1$. The unperturbed trajectories of the saddle-loop at $z_1 = -\frac{3}{2}\nu_1$ can be obtained as

$$z_1(t,t_o) = -\frac{3\nu_1}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{\nu_1}}{2}\,(t-t_o)\right],$$

$$z_2(t,t_o) = \frac{3\nu_1}{2}^{3/2}\mathrm{sech}^2\left[\frac{\sqrt{\nu_1}}{2}\,(t-t_o)\right]\tanh\left[\frac{\sqrt{\nu_1}}{2}\,(t-t_o)\right]. \tag{11}$$

Following the Melnikov procedure given in Guckenheimer and Holmes[7] for the perturbed autonomous system (10) (with $\varepsilon \neq 0$), we obtain the condition that

$$\int_{-\infty}^{\infty} v^2(t,t_o)\,[\nu_2 + u(t,t_o)]\,dt = 0 \tag{12}$$

for the saddle connection to not break under perturbation. Eq. (12) may be written as

$$\frac{27}{4}(\nu_1)^{7/2}(I_1 - I_2) = 0$$

where $$I_1 = \frac{2\nu_2}{3\nu_1}\int_{-\infty}^{\infty} \operatorname{sech}^4\xi \tanh^2\xi d\xi = \frac{8\nu_2}{45\nu_1},$$

$$I_2 = \int_{-\infty}^{\infty} \operatorname{sech}^6\xi \tanh^2\xi d\xi = \frac{16}{105}$$

Thus the saddle connection is preserved when

$$\nu_2 = \frac{6}{7}\nu_1 \quad \text{or} \quad \alpha_2 = \frac{6}{7}\alpha_1.$$

It can be concluded that there exist two saddle connections: one at $\mu_2 = \frac{6}{7}\mu_1$ passing through the trivial solution, and one at $\mu_2 = \frac{1}{7}\mu_1$ that passes through the nontrivial solution as shown in Figure 1.

The above calculations indicate the existence of a limit cycle in the regions 3 and 8 in Fig. 2. The uniqueness of this limit cycle will be demonstrated following the procedure outlined in Chow and Hale[8] and Carr et. al[9]. Every limit cycle within the saddle-loop must encircle the equilibrium point $(-\nu_1,0)$ crossing the x axis between $-\nu_1$ and 0 at (b,0). Let the other crossing point be (c,0). The limit cycles for the perturbed system is denoted as $\Gamma_\varepsilon(b,\nu_1,\nu_2)$. Along the solution of Eq. (10) we have

$$\dot{H}(z_1,z_2) = \varepsilon z_2^2(\nu_2 + z_1)$$

and since $\Gamma_\varepsilon(b,\nu_1,\nu_2)$ is a limit cycle we have

$$\int_{\Gamma_\varepsilon} \dot{H}dt = 0, \text{ i.e., } F(b,\varepsilon,\nu_1,\nu_2) = \int_{\Gamma_\varepsilon} z_1^2(\nu_2 + z_1)dt = 0$$

The function $F(b,0,\nu_1,\nu_2)$ may be written explicitly as

$$F(b,0,\nu_1,\nu_2) = \nu_2\tilde{J}_0(b,\nu_1) + \tilde{J}_1(b,\nu_1) \tag{13}$$

where

$$\tilde{J}_o(b,\nu_1) = \int_{\Gamma_o} z_2^2 dt \quad , \quad \tilde{J}_1(b,\nu_1) = \int_{\Gamma_o} z_1 z_2^2 dt$$

Thus, the solution of $F(b,0,\nu_1,\nu_2) = 0$ is given by

$$\nu_2 = - \tilde{J}_1(b,\nu_1) \; / \; \tilde{J}_o(b,\nu_1) \; .$$

Differentiating (13) yields

$$\frac{\partial F}{\partial \nu_2}(b,0,\nu_1,\nu_2) = \tilde{J}_o(b,\nu_1) \neq 0$$

which implies, by the implicit function theorem (IFT) that there exists a unique continuously differentiable function $\nu^*(b,\varepsilon,\nu_1)$ such that $F(b,\varepsilon,\nu_1,\nu^*(b,\varepsilon,\nu_1)) = 0$ for sufficiently small ε and

$$\nu^*(b,o,\nu_1) = - \tilde{J}_1(b,\nu_1) \; / \; \tilde{J}_o(b,\nu_1)$$

Having shown the existence of a limit cycle by IFT, we proceed to show that the limit cycle is unique for a given value of ν_1 and ν_2 by demonstrating that ν^* is monotonic in b. However, it will be more convenient to employ in place of b another parameter h, which corresponds to the energy level i.e.

$$h = H(b,o) = - \nu_1 \frac{b^2}{2} - \frac{b^3}{3}$$

This change of parameter is justified, since $dh/db = - b(\nu_1 + b) > 0$ for $- \nu_1 < b < 0$. Thus

$$\nu_2 = - J_1(h) \; / \; J_o(h) = - P(h) \tag{14}$$

where $J_o(h) = \tilde{J}_o(b(h),\nu_1)$, $J_1(h) = \tilde{J}_1(b(h),\nu_1)$ and the dependence of ν_1 is suppressed. Since $z_2(b(h)) = z_2(c(h)) = 0$, it can be verified that

$$J_o'(h) = \int_{b(h)}^{c(h)} \frac{dz_1}{z_2} \quad , \quad J_1'(h) = \int_{b(h)}^{c(h)} \frac{z_1}{z_2}\, dz_1$$

Furthermore, the limits

$$\lim_{h \to 0} P(h) = -(6/7)\nu_1 \text{ and } \lim_{h \to -\nu_1^3/6} P(h) = \lim_{h \to -\nu_1^3/6} \frac{J_1'(h)}{J_o'(h)} = -\nu_1$$

agree with the previous calculations of saddle-loop and Hopf bifurcations. The following relationships between $J_o(h)$, $J_1(h)$ and their derivatives can be obtained using the expression for z_2:

$$J_o(h) = \nu_1^2\, J_1'(h) = \int_{b(h)}^{c(h)} \frac{z_1^3}{z_2}\, dz_1$$

$$5J_o(h) - 6h\, J_o'(h) + \nu_1^2 J_1'(h) = 0$$

$$35J_1(h) + 6\left[h\nu_1 J_o'(h) - \left(\nu_1^3 + 5h\right)J_1'(h)\right] = 0 \qquad (15)$$

$$\left(\nu_1^3 + 6h\right) J_1''(h) = J_o'(h)\nu_1 + J_1'(h)$$

$$6h\left(\nu_1^3 + 6h\right) J_o'' = \nu_1^2 J_1'(h) - 6h\, J_o'(h)$$

Now using the above relations one can show that if $P'(h_1) = 0$ for some $h_1\ \varepsilon\left(-\nu_1^3/6, 0\right)$ then

$$6h_1\left(\nu_1^3 + 6h_1\right) \frac{P''(h_1)J_o(h_1)}{J_o'(h_1)} = -\left(\nu_1(P(h_1)) - \frac{6h_1}{\nu_1}\right)^2 + \frac{6h_1}{\nu_1^2}\left(\nu_1^3 + 6h_1\right) < 0$$

$$7\nu_1^2 P^2(h_1) + 6(\nu_1^3 - 2h_1)P(h_1) - 6h_1\nu_1 = 0 \qquad (16)$$

Since $6h_1(\nu_1^3 + 6h_1) < 0$ and $J_o(h_1) / J_o'(h_1) < 0$, it follows from the inequality (16a) that $P''(h_1) > 0$. Furthermore, it follows from Eq. (16b) that $-\nu_1 < P(h_1) < 0$. In other words, if there is a point h_1 for which $p'(h_1) = 0$, then the function P is concave up at this point with the value of the function at this point lying between $-\nu_1$ and 0. Since the end points of $p(h_1)$ are at $-\nu_1$ and $-6/7\ \nu_1$, $p'(h) \neq 0$ for $h_1 \epsilon (-\nu_1^3/6, 0)$, in fact $p'(h) > 0$. Thus, $p(h_1)$ is a monotonically increasing function implying a unique limit cycle.

V. Discussion of Results and Conclusion

The results of this analysis are illustrated in Fig. 2, where the space of unfolding parameters is divided into ten regions indicating the various bifurcations and phase portraits of Eq. (5). In passing from region one to region two along OS_1, the nontrivial fixed point changes from an unstable node to an unstable focus while the trivial solution remains a saddle node. Along OH_1, the nontrivial fixed point undergoes a Hopf bifurcation giving birth to an unstable limit cycle. It has been shown that this limit cycle is unique and disappears along OL_1 due to a global bifurcation and a saddle loop that passes through the trivial fixed point is produced. The nontrivial fixed point, in passing from region four to region five along OS_1', changes from a stable focus to a stable node while the trivial fixed point remains a saddle node. Along OT a transcritical bifurcation takes places where an exchange of stability between the trivial and nontrivial fixed points occurs. Finally, in going from region six through to region ten the nontrivial fixed point remains a saddle node while the scenario of bifurcations for the trivial solution is similar to that of the nontrivial fixed point detailed above and presented in Figure 2.

In this note a complete unfolding of a co-dimension two bifurcation due

to a double zero eigenvalue of the equations of pitching motion of an aircraft was carried out in the vicinity of zero stiffness derivative $S(\sigma_c, h_c) = 0$, and zero damping derivative $D(\sigma_c, h_c) = 0$. Unfolding of such a singularity will uncover all possible bifurcations that may be present in the vicinity of the singularity, in addition to the results of Hui and Tobak[1]. Even though the problem considered is not rich enough to fully demonstrate the method of unfolding of a co-dimension two bifurcation point, as most of the local results could have been obtained using methods adopted in Ref. 1, this method, nevertheless, provides the results pertaining to uniqueness of limit cycles and global bifurcations.

M_∞	σ_c	h_c	$-(\partial S/\partial\sigma)$	$-(\partial D/\partial\sigma)$	$-(\partial^2 S/\partial^2\sigma)$	$-(\partial^2 D/\partial^2\sigma)$
2.0000	12.3886	0.4432	0.3158	1.5078	10.5616	25.1282
2.5000	19.2534	0.4471	0.1738	1.6417	9.5587	29.1733
3.0000	23.7198	0.4524	0.1240	1.7658	9.9810	33.3559
3.5000	26.7177	0.4571	0.1166	1.8872	10.8663	37.2688
4.0000	28.7890	0.4608	0.1233	1.9906	11.7336	40.5696
5.0000	31.3435	0.4656	0.1408	2.1382	12.9768	45.3055
6.0000	32.7784	0.4684	0.1506	2.2277	13.6656	48.2253
7.0000	33.6578	0.4701	0.1543	2.2831	14.0607	50.0638
8.0000	34.2344	0.4713	0.1557	2.3190	14.3154	51.2766
9.0000	34.6325	0.4721	0.1565	2.3433	14.4953	52.1131
10.000	34.9187	0.4727	0.1571	2.3604	14.6224	52.7144

Table 1. Critical Parameter Values Associated with Double Zero Eigenvalue.

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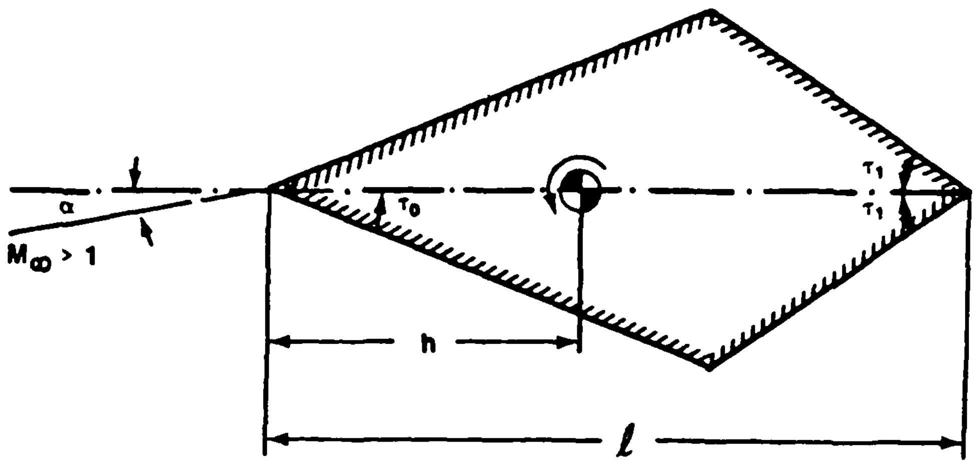

Aerofoil at Angle of Attack α

Figure 1 Thin double-wedge aerofoil

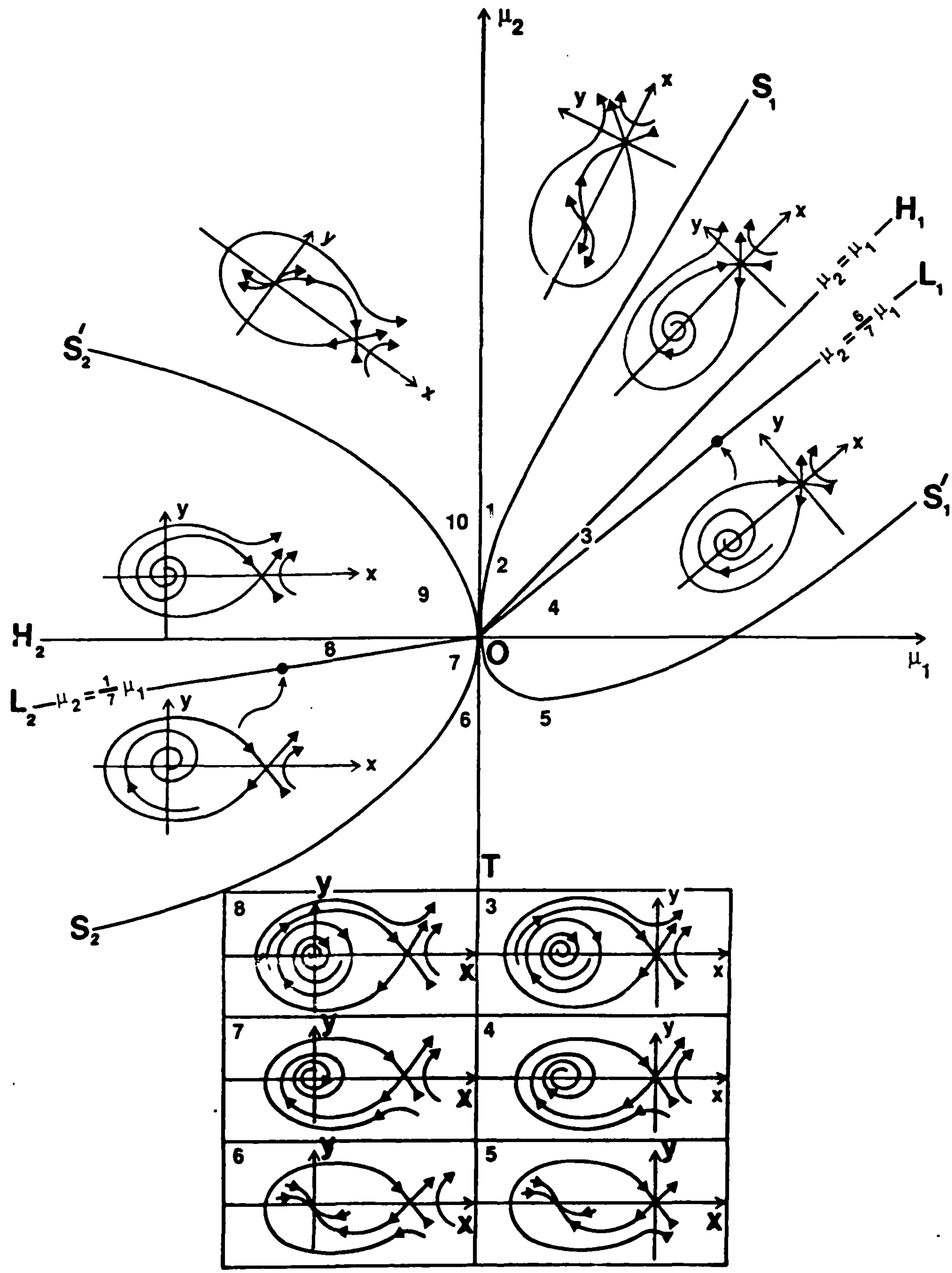

Figure 2 Various bifurcations and phase portraits of Equation 5

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Unfolding of Degenerate Hopf Bifurcation for Supersonic Flow past a Pitching Wedge

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This paper investigates the stability and bifurcation behavior of a double-wedge aerofoil performing a pitching motion at high angles of attack. When a pair of complex conjugate eigenvalues crosses the imaginary axis of the eigenvalue plane, the trivial solution loses stability giving rise to a periodic solution, known as Hopf bifurcation, provided certain transversality conditions are not violated. The existence of degenerate Hopf bifurcation due to the violation of Hopf's transversality condition at certain critical values of the system parameters is shown. The behavior of the pitching motion near these critical values is examined by unfolding the degeneracies. For the supersonic double-wedge aerofoil, various parameters defining the bifurcation paths were numerically evaluated.

I. Introduction

In recent years several new mathematical ideas have influenced the study of stability and bifurcation phenomena of nonlinear dynamical systems. In this paper, aerodynamic stability of a double-wedge subject to a single degree of freedom pitching motion is investigated. Recently Hui & Tobak[1] analyzed the Hopf bifurcation that results when a steady flight becomes unstable by increasing the angle of attack σ beyond a critical value σ_c, holding all other flow parameters fixed. If more than one parameter is allowed to vary, such as angle of attack σ and pivot position h, then phenomena other than simple Hopf bifurcation may occur. For the case of a double-wedge, it is found that if both angle of attack σ and pivot position h reach certain critical values σ_c and h_c, respectively, then the transversality condition of the Hopf bifurcation theorem does not hold and a so-called degenerate Hopf bifurcation takes place.[2] However, this degenerate phenomenon is nongeneric. In order to more completely understand the behavior of the system, it is useful to examine it near the singularities $\sigma=\sigma_c$ and $h=h_c$ by either incorporating an unfolding parameter or by studying the problem as a multiple parameter system.

In this paper, the former approach will be used to understand the bifurcation behavior of the system. A general framework for unfolding such degeneracies has been given by Golubitsky and Langford[3] using the singularity theory.

II. Statement of the Problem

Consider an aircraft in flight free to undergo a single degree of freedom pitching motion. The equations of pitching motion can be expressed as

$$\frac{d\alpha}{dt}=\dot{\alpha}, \qquad I\frac{d\dot{\alpha}}{dt}=M(t) \tag{1}$$

where α is the angle of attack of the steady flight, I is the moment of inertia of the vehicle about the pivot axis, and $M(t)$ is the pitching moment at instantaneous time t of the aerodynamic forces about the same axis. When the motion is slowly varying,[4] the pitching moment $M(t)$ may be characterized with sufficient accuracy by the instantaneous angle of attack $\alpha(t)$ and the instantaneous rate of change of the angle of attack $\dot{\alpha}(t)$. Supose $\alpha=\sigma$ is an equilibrium state of the system of Eqs. (1); then, putting $\alpha(t)=\sigma+\psi(t)$, the variational equations about the equilibrium position can be written as

$$\frac{d\psi}{dt}=\dot{\psi}$$

$$\frac{d\dot{\psi}}{dt}=\frac{M}{I}(t)=\frac{1}{2I}\rho_\infty V_\infty^2 SL\,[C_m(0,0,\sigma,h)-C_m(\psi,\dot{\psi},\sigma,h)] \tag{2}$$

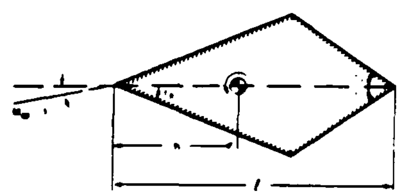

Fig. 1a Aerofoil at angle of attack.

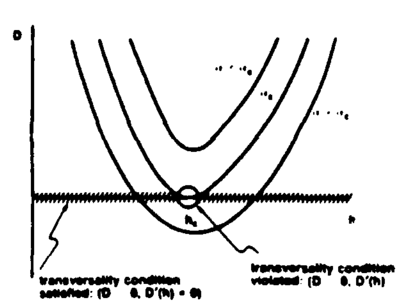

Fig. 1b Transversality condition and its violation.

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where ψ is the angular displacement of motion measured from the angle of attack σ of the steady flight; ρ_∞ and V_∞ are the freestream density and velocity, respectively; $\bar{S}$ and L are the reference area and length; and h represents the distance between the apex and the pivot position as defined in Fig. 1a. The function $C_m(\psi,\dot{\psi},\sigma,h)$ represents the pitching moment coefficient of the aerodynamic forces about the pivot axis and $C_m(0,0,\sigma,h)$ is its steady value at a fixed angle of attack σ. Even though C_m depends on the flight Mach number M_∞, the specific heats of the air and the aircraft shape, these parameters will be considered as "passive" parameters in this analysis. For a finite amplitude, slow, periodic, pitching motion with angular displacement $\psi(t)$ around a mean angle of attack σ, with terms of $O(\dot{\psi}^2,\ddot{\psi})$ assumed negligible, we can write[1]

$$-C_m(\psi,\dot{\psi},\sigma,h) = f(\sigma+\psi,h) + g(\sigma+\psi,h)\dot{\psi}$$

which reduces the second of Eqs. (2) to

$$\frac{d\dot{\psi}}{dt} = F(\psi,\dot{\psi},\sigma,h)$$

where

$$F(\psi,\dot{\psi},\sigma,h) = \frac{M(t)}{I} = \kappa[f(\sigma+\psi,h) - f(\sigma,h)$$

$$+ g(\sigma+\psi,h)\dot{\psi}], \qquad \kappa = \frac{1}{2I}\rho_\infty V_\infty^2 \bar{S}L$$

Equations (2) represent a pair of autonomous differential equations in R^2 the trivial solution of which is $\psi = 0$. The objective of this investigation is to understand the stability of this trivial solution and the bifurcation behavior of Eqs. (2) as the system parameters σ and h are varied.

III. Stability of the Trivial Solution

The functions $f(\sigma,h)$ and $g(\sigma,h)$ are related to the stiffness derivative $S(\sigma,h)$ and the damping derivative $D(\sigma,h)$ of classical aerodynamics as follows:

$$\kappa\frac{\partial f}{\partial \sigma}(\sigma,h) = -S(\sigma,h), \qquad \kappa g(\sigma,h) = -D(\sigma,h)$$

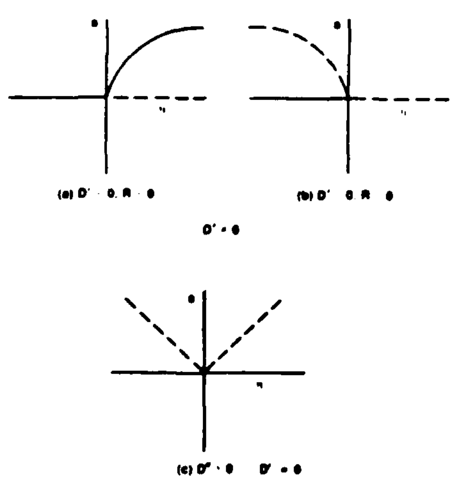

Fig. 2 Bifurcation diagrams: a) supercritical, b) subcritical, and c) degenerate Hopf bifurcation.

Introducing new state variables $y_1 = \psi$, $y_2 = \dot{\psi}$, Eqs. (2) may be written in the form

$$\dot{y} = Ay + \begin{bmatrix} 0 \\ F(y_1,y_2,\sigma,h) \end{bmatrix} + O(|y|^4) \tag{3}$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A(\sigma,h) = \begin{bmatrix} 0 & 1 \\ -S(\sigma,h) & -D(\sigma,h) \end{bmatrix}$$

$$F(y_1,y_2,\sigma,h) = \hat{B}_{11}y_1^2 + \hat{B}_{12}y_1y_2 + \hat{C}_{111}y_1^3 + \hat{C}_{112}y_1^2y_2$$

$$\hat{B}_{11} = -\frac{1}{2}\frac{\partial S}{\partial \sigma}(\sigma,h), \qquad \hat{B}_{12} = -\frac{1}{2}\frac{\partial D}{\partial \sigma}(\sigma,h)$$

$$\hat{C}_{111} = -\frac{1}{3!}\frac{\partial^2 S}{\partial \sigma^2}(\sigma,h), \qquad \hat{C}_{112} = -\frac{1}{2}\frac{\partial^2 D}{\partial \sigma^2}(\sigma,h)$$

The stability of the trivial solution is governed by the eigenvalues of the matrix A, which are

$$\lambda = -\frac{D}{2} \pm i\sqrt{S - D^2/4} = -\frac{D}{2} \pm i\omega \tag{4}$$

It is evident that the equilibrium position is asymptotically stable when

$$S(\sigma,h) > 0, \qquad D(\sigma,h) > 0$$

and instability occurs when $D(\sigma,h) = 0$ and $S(\sigma,h) > 0$, giving rise to a pair of pure imaginary eigenvalues; or when $D(\sigma,h) > 0$ and $S(\sigma,h) = 0$, giving rise to a zero and a negative eigenvalue; and the nongeneric case $D(\sigma,h) = 0$ and $S(\sigma,h) = 0$, giving rise to a double zero eigenvalue. Only the first case will be considered. Though the extension of the general results obtained in this paper for a two parameter system is possible, we shall analyze the problem as if it were a one parameter system. To avoid duplication of calculations, we shall refer to the bifurcation parameter as μ which can represent the angle of attack σ (or the pivot position h) holding h (or σ) constant. Let us assume that at $\mu = \mu_c$, the damping derivative becomes zero $D(\mu_c) = 0$, the stiffness derivative $S(\mu_c) > 0$, and the corresponding eigenvalues are $\lambda_c = \pm i$ and $\omega_c = \pm i\sqrt{S(\mu_c)}$. According to Hopf's theorem,[4] the system described by Eq. (3), along with the conditions

$$\omega(\mu_c) = \omega_0 > 0, \qquad D(\mu_c) = 0 \tag{5a}$$

$$\left.\frac{dD}{d\mu}\right|_{\mu=\mu_c} = D'(\mu_c) \neq 0 \tag{5b}$$

has a family of periodic solutions bifurcating out of the equilibrium solution $y = 0$, parameterized by the amplitude a for $|a|$ small. Furthermore, Hopf showed that along the periodic solution branch μ is an even function of a given by

$$\mu = \mu_c + \mu_2 a^2 + \mu_4 a^4 + \ldots \tag{6a}$$

assuming

$$\mu_2 \neq 0 \tag{6b}$$

These solutions exist either for $\mu > \mu_c$ (*supercritical* Hopf bifurcation) or for $\mu < \mu_c$ (*subcritical* Hopf bifurcation) depending on the sign of μ_2. Bifurcation of such periodic solutions out of the trivial solution, when Hopf's conditions, viz., Eqs. (5b) or (6b) or both Eqs. (5b) and (6b), are violated is in

general called *degenerate* Hopf bifurcation. The preceding analysis holds for any single degree of freedom motion. Application of the analysis requires a knowledge of the stiffness derivative S and the damping derivative D together with their partial derivatives. The stiffness and damping derivatives for a double-wedge aerofoil in supersonic flow have been determined by Hui.[6] In this paper their partial derivatives have been calculated numerically using the results of Ref. 6. For the problem of double-wedge aerofoil it is the violation of Eq. (5b) which occurs, hence it is the degenerate Hopf bifurcation associated with the violation of Eq. (5b) which will be studied. Hopf's transversality condition, as well as its violation when h is taken as the bifurcation parameter, is shown in Fig. 1b.

IV. Bifurcation Analysis

In this section both Hopf and degenerate Hopf bifurcation will be considered. Assume that at $\mu = \mu_c$, the damping derivative becomes zero $[D(\mu_c) = 0]$ and the stiffness derivative is positive $[S(\mu_c) > 0]$. The eigenvalue is $\lambda_c = \pm i\omega_c = \pm i\sqrt{S(\mu_c)}$ and the corresponding eigenvector is $(1, \lambda_c)$.

To study the Hopf bifurcation and its stability, a change of coordinates is made to put the system of Eqs. (2) into a standard form. This is achieved by the linear transformation

$$y = Tx \tag{7}$$

where

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \omega_c \end{bmatrix}$$

is the matrix consisting of the real and imaginary parts of the critical eigenvalue and $x = (x_1, x_2)$ represents the new state variables. The above transformation yields the system of equations with the linear part in standard form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega_c & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\frac{B_{11}}{\omega_c} x_1^2 + B_{12} x_1 x_2 + \frac{{}^0C_{111}}{\omega_c} x_1^3 + C_{112} x_1^2 x_2 \right] \tag{8}$$

Before proceeding to degenerate Hopf bifurcation, a summary of the results for the regular Hopf bifurcation will be given.

Hopf Bifurcation

Now the formulas for Hopf bifurcation given by Guckenheimer and Holmes[3] and Ariaratnam and Sri Namachchivaya,[2] are used to obtain the equation governing the bifurcating path

$$2Ra^3 - D'(\mu_c)\eta a = 0 \tag{9}$$

where

$$R = \frac{1}{8\omega_c^2}(B_{11}B_{12} + \omega_c^2 C_{112}), \quad \eta = \mu - \mu_c \ll 1,$$

and a represents the amplitude of the bifurcating periodic solution given by

$$x_1 = a\sin\phi + \frac{a^2}{\omega_c}\left[\frac{1}{2}\frac{B_{11}}{\omega_c} - \frac{1}{3}B_{12}\sin 2\phi + \frac{1}{6}\frac{B_{11}}{\omega_c}\cos 2\phi\right]$$

$$x_2 = a\cos\phi - \frac{a^2}{\omega_c}\left[\frac{1}{3}\frac{B_{11}}{\omega_c}\sin 2\phi + \frac{2}{3}B_{12}\cos 2\phi\right] \tag{10}$$

where

$$\phi = \bar{\omega}t + \text{const} \qquad \bar{\omega} = \omega_c + a^2\left(P + \frac{\omega'}{D'}R\right)$$

$$P = \frac{1}{\omega_c^3}\left[\frac{1}{3}B_{11}^2 - \frac{1}{6}\omega_c^2 B_{12}^2 - \frac{3}{8}\omega_c^2 C_{111}\right]$$

The amplitude parameter relationship can be written using Eq. (9) as

$$\eta = \frac{2R}{D'(\mu_c)}a^2 \tag{11}$$

provided that $D'(\mu_c) \neq 0$.

When $D'(\mu) < 0$, which is generally the case when eigenvalues cross from left to right in the complex λ-plane, it is evident from Eq. (11) that the bifurcating path exists for $\eta > 0$ only if $R < 0$ (*supercritical* bifurcation) as shown in Fig. 2a. Similarly the bifurcation path exists for $\eta < 0$ only if $R > 0$ (*subcritical* bifurcation) as shown in Fig. 2b. The opposite is true for $D'(\mu_c) > 0$. It is well know[n t]hat the damping and the stiffness derivatives are respective[ly qu]adratic and linear in h, i.e.,

$$D(\sigma, h) = D_0(\sigma) + D_1(\sigma)h + D_2(\sigma)h^2$$

$$S(\sigma, h) = S_0(\sigma) + S_1(\sigma)h \tag{12}$$

Furthermore, the qualitative variations of the quantities $D(\sigma,h)$ and $S(\sigma,h)$ with σ and h can be found in Hui[6] for double-wedge aerofoil. By considering σ as the bifurcation parameter, i.e., $\mu = \sigma$, the results of Hui and Tobak[1] are recovered. Similarly, the amplitude parameter relationship, considering h as the bifurcation parameter, can be written as

$$h - h_c = \frac{l}{8(D_1(\sigma) + 2D_2(\sigma)h_c)}\left\{S\frac{\partial}{\partial\sigma}\left(\frac{\partial D/\partial\sigma}{S}\right)\right\}_{h=h_c} \tag{13}$$

Degenerate Hopf Bifurcation

Now we shall examine the bifurcations that can take place when Hopf's transversality condition [Eq. (5b)] is violated, i.e., degenerate Hopf bifurcation. It can be shown that in double-wedge and flat-plate aerofoils, degeneracies of the above-mentioned type for both parameters ($\partial D/\partial\sigma = 0$, $\partial D/\partial h = 0$) are present. However, $S(\sigma,h) > 0$ only for the second case, and thus the degenerate Hopf bifurcation when $D = 0$, $\partial D/\partial h = 0$, will be examined, i.e., when

$$D_1^2(\sigma_c) = 4D_0(\sigma_c)D_2(\sigma_c), \qquad h_c = -\frac{D_1(\sigma_c)}{2D_2(\sigma_c)}$$

provided $D_2(\sigma_c) \neq 0$. Violation of the transversality condition when h is considered as the bifurcation parameter is shown in Fig. 1b. Furthermore, for the eigenvalues to be purely imaginary we should have

$$S_0(\sigma_c) - \frac{S_1(\sigma_c)D_1(\sigma_c)}{2D_2(\sigma_c)} > 0$$

Since we are studying the local behavior of the system, Eq. (8), as opposed to the global one, subsequent analysis is performed in small neighborhood of x, while the above conditions prevail. Thus, making use of the general results given in Ref. 5 for degenerate Hopf bifurcation, the equations governing the bifurcating path and improved frequency for the wedge problems are obtained as:

$$2Ra^3 - D_2(\sigma_c)\bar{h}^2 a = 0 \tag{14a}$$

and

$$\bar{\omega} = \omega_c + a\left\{a\left[P - \frac{S_1^2(\sigma_c)R}{4\omega_c^2 D_2(\sigma_c)}\right] \pm \frac{S_1(\sigma_c)}{\omega_c}\left(\frac{R}{2D_2(\sigma_c)}\right)\right\} \quad (14b)$$

respectively, where $\bar{h} = h - h_c$. The existence of the bifurcating path depends on the sign of $R/D_2(\sigma_c)$. In other words, a bifurcating solution exists only if $R/D_2(\sigma_c) > 0$. Therefore, in degenerate Hopf bifurcation, the bifurcating path exists on *both* sides of the a axis as opposed to Hopf bifurcation where the bifurcating path exists *either* for $\bar{h} > 0$ or for $\bar{h} < 0$. The bifurcating path can be expressed explicitly as

$$h - h_c = \pm \frac{a_0}{2\sqrt{D_2(\sigma)}}\left\{S\frac{\partial}{\partial\sigma}\left(\frac{\partial D/\partial\sigma}{S}\right)\right\}_{h=h_c}^{\frac{1}{2}} \quad (15)$$

It may be noted that each bifurcating path defined by Eq. (15) has a distinct frequency given by Eq. (14b). If $D_2 > 0$, then both bifurcating paths given in Eq. (15) are unstable while the trivial solution is stable. On the other hand, if $D_2 < 0$ no bifurcating solution exists. These results are shown in Fig. 2c.

V. Unfolding

Now to consider the behavior of the system near this nongeneric degenerate Hopf bifurcation, an unfolding parameter is introduced. Since the degeneracy occurs while considering h as a bifurcation parameter, it is natural to consider σ as an unfolding parameter. Loosely speaking, a parameter is said to be an unfolding parameter when it fills in the missing lower order term in the bifurcation equation. The main theoretical results classifying various bifurcations and their unfoldings when the conditions of Eq. (5b) or (6b) or both fail were presented by Golubitsky and Langford[3] using singularity theory. Making use of available results,[3] the equation governing the bifurcating paths incorporating $\partial D/\partial\sigma$ can be written as

$$2Ra^3 - \left(\frac{\partial^2 D}{\partial h^2}\bigg|_{\sigma_c,h_c}\bar{h}^2 + \frac{\partial D}{\partial\sigma}\bigg|_{\sigma_c,h_c}\bar{\sigma}\right)a = 0 \quad (16)$$

which simplifies to

$$a^2 - \frac{D_2(\sigma_c)}{R}\bar{h}^2 - \beta = 0 \quad (17)$$

where

$$\beta = \frac{1}{2R}[D_0'(\sigma_c) + D_1'(\sigma_c)h_c + D_2'(\sigma_c)h_c^2]\bar{\sigma}$$

$$\bar{\sigma} = \sigma - \sigma_c$$

$$\bar{h} = h - h_c$$

Depending on the sign of $D_2(\sigma_c)/R$ and β, a set of bifurcation diagrams as shown in Fig. 3 can be obtained. For $\beta = 0$, Eq. (17) yields the previously obtained result of Eq. (15). In Fig. 3a the case $D_2(\sigma_c)/R < 0$ is considered, for which Eq. (17) represents an ellipse for $\beta > 0$ and has no real solution for $\beta < 0$. On the other hand, for $D_2(\sigma_c)/R > 0$ Eq. (17) represents an hyperbola for $\beta \neq 0$ as sketched in Fig. 3b where s and u indicate stable and unstable solutions, respectively.

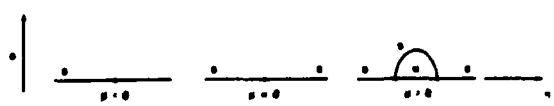

Fig. 3a Case $D_2(\sigma_c)/R < 0$.

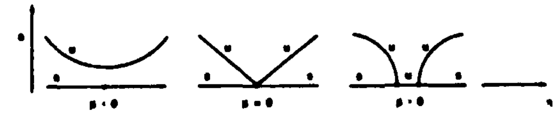

Fig. 3b Case $D_2(\sigma_c)/R > 0$.

Fig. 3 Unfoldings.

Applying the results obtained in the above analysis to Hui's solution[6] for a double-wedge aerofoil, it is found that the case corresponding to Fig. 3b occurs. The special cases of a flat plate aerofoil and a wedge may be obtained from Hui's solution[6] by an appropriate choice of shape parameters τ_0, τ_1 in Fig. 1a. For the purposes of this study, we focus our attention on the case $\tau_0 = \tau_1 = 5°$ since this approximates a thin aerofoil. For a double-wedge in supersonic flight the various components of the stiffness and damping derivatives, namely S_0, S_1, D_0, D_1, and D_2 are plotted in Figs. 4 and 5. Using Eq. (12), $S(\sigma,h)$ and $D(\sigma,h)$ for a given value of σ and h may be obtained.

In addition to these results, various other quantities required for the bifurcation analysis and unfolding are also calculated and displayed in Figs. 6 to 8. In Fig. 6 the relationships between h_c and σ_c and between M_∞ and σ_c are plotted. From this figure one may obtain the critical value of h and σ

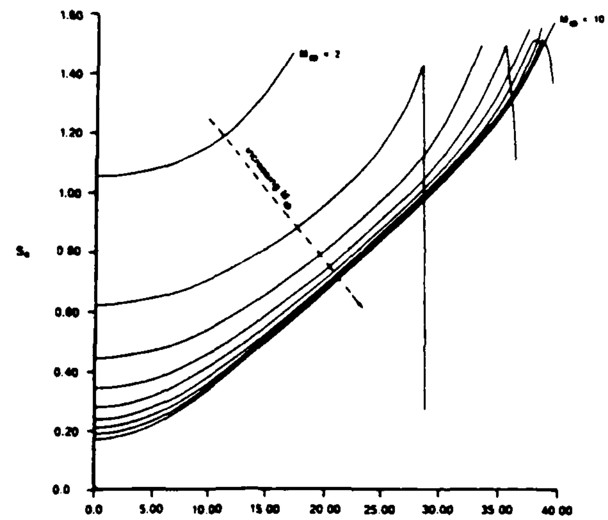

Fig. 4a S_0 vs σ for $M_\infty = 2,3,\ldots,10$.

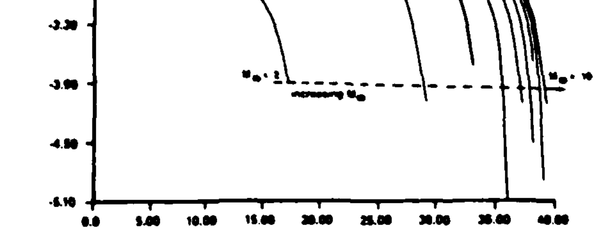

Fig. 4b S_1 vs σ for $M_\infty = 2,3,\ldots,10$.

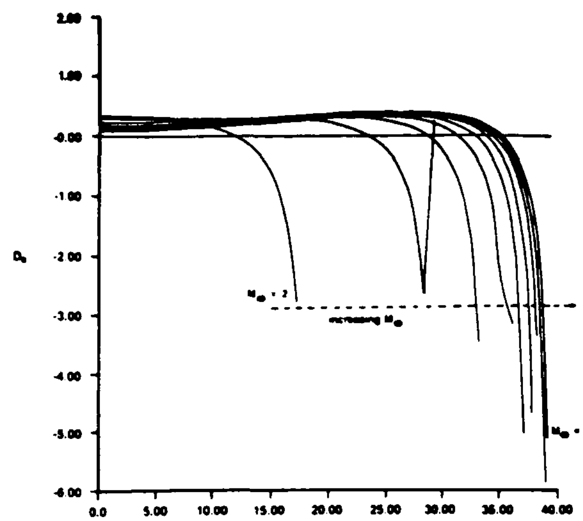

Fig. 5a D_0 vs σ for $M_\infty = 2,3,\ldots,10$.

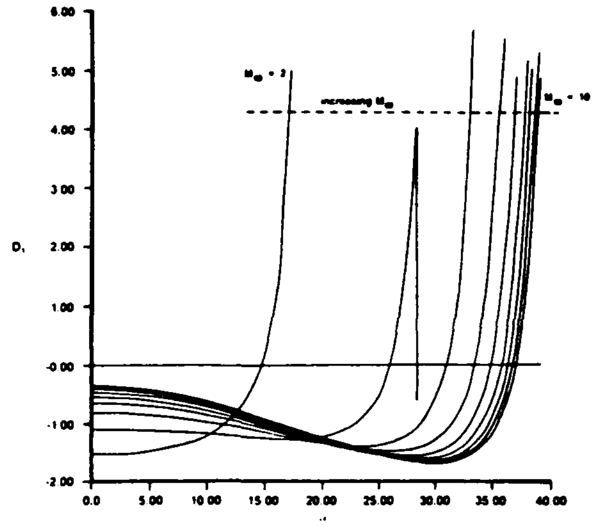

Fig. 5b D_1 vs σ for $M_\infty = 2,3,\ldots,10$.

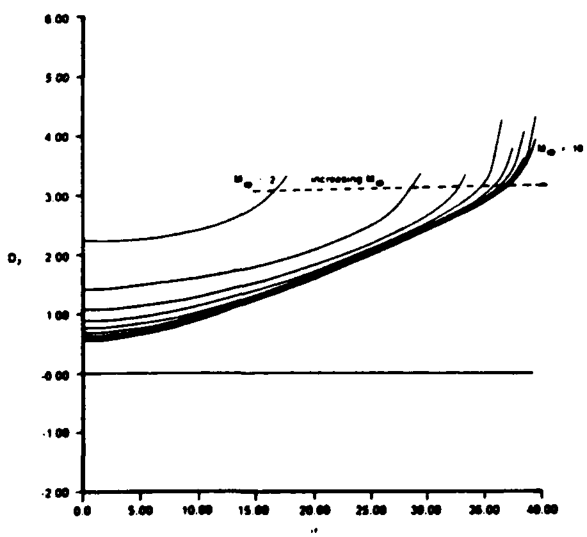

Fig. 5c D_2 vs σ for $M_\infty = 2,3,\ldots,10$.

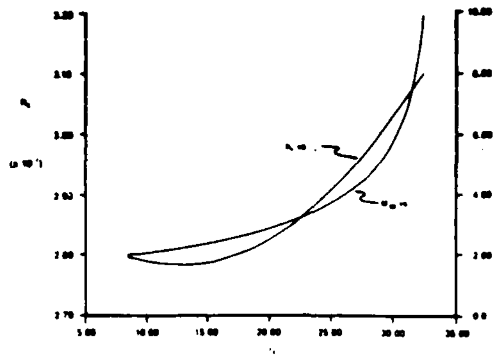

Fig. 6 h_c vs σ_c superimposed on M_∞ vs σ_c.

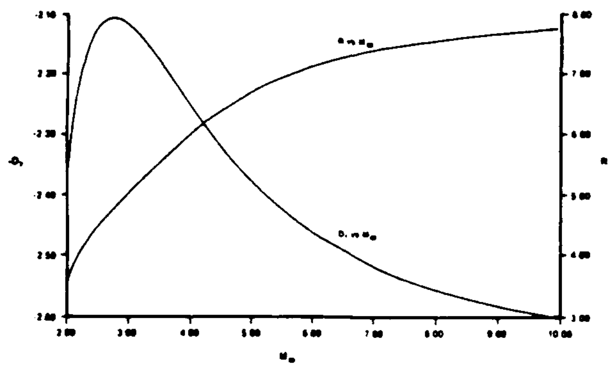

Fig. 7 R vs M_∞ superimposed on $-D_2$ vs M_∞.

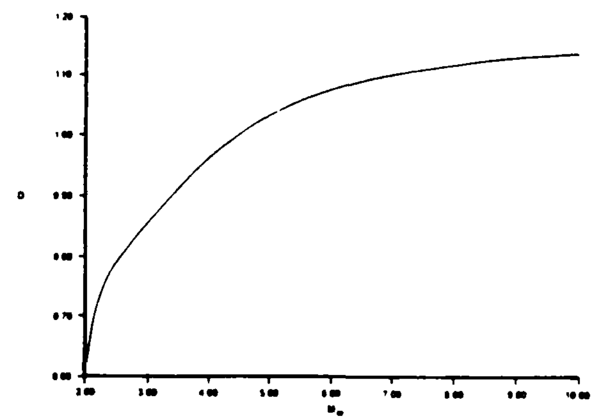

Fig. 8 D' vs M_∞.

for a specific Mach number. In Fig. 7 the relationship between the bifurcating coefficient R and M_∞ is superimposed on the relationship between the second derivative of the real part of the eigenvalue with respect to h (i.e., $-D_2$), and M_∞. With the help of these figures the bifurcating path given by Eq. (15) can be obtained. The relationship between the unfolding parameter $2R\beta/\delta = (\partial D/\partial \sigma)(\sigma_c, h_c)$ and M_∞ is plotted in Fig. 8. Therefore, Figs. 6 to 8 contain all the information required to completely determine the various bifurcations that can take place.

VI. Conclusion

In this paper, the aerodynamic stability and bifurcation of an aerofoil subject to a single degree of freedom pitching mo-

tion has been studied. It was found that, in addition to the simple Hopf bifurcation, degenerate Hopf bifurcation can take place if more than one parameter is allowed to vary. Furthermore, it was shown that in the degenerate case, there will be two periodic bifurcating paths (on both sides of the a axis), with two different frequencies. These frequencies are either both stable or both unstable, as opposed to Hopf bifurcation where the bifurcating path exists either for $\eta > 0$ or for $\eta < 0$. However, the situation giving rise to degenerate Hopf bifurcation is nongeneric. By the introduction of an unfolding parameter, the possible generic bifurcations that can take place near the singularity were obtained. This reveals that for $D_2(\sigma_c)/R > 0$, there exist either two subcritical bifurcations or no bifurcation in a neighborhood of the degeneracy depending upon the sign of β. Similarly, it was found that for $D_2(\sigma_c)/R < 0$ there exist either two supercritical bifurcations or no real solutions in a neighborhood of the degeneracy depending upon the sign of β. In addition, numerical results of the various components of the stiffness and damping derivatives, and other quantities required for the bifurcation analysis, were presented for a thin aerofoil.

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APPENDIX D

Introduction

In the last fifty years fighter aircraft have evolved from designs characterized by subsonic speeds, moderate to high aspect ratios and negligible sweep to designs capable of supersonic speeds employing low aspect ratio wings with significant sweep and taper. Many of these changes were dictated by the need to reduce drag in transonic and supersonic flight and this in turn led to new problems in finding a suitable compromise between performance, handling qualities and stability/control requirements. Recent developments have demonstrated that an aircraft with reduced static stability (RSS) supplemented with an active flight control system (ACFS) results in lower weight and increased maneuverability [1]. The design of such aircraft requires that the control systems and the aerodynamic configuration be considered together from the start. Traditionally flight controls were used only to improve the flying qualities of a chosen configuration. The feasibility of this control-configured vehicle (CCV) design approach is amply proven by the X-29 research program [2].

The CCV approach has two immediate implications for combat aircraft design. The first is the possibility of direct-force maneuvers (DFM). Direct force maneuvers refer to the "ability of the aircraft to yaw and pitch independently of the flight path or to maneuver at constant fuselage orientation". This is especially significant in target tracking. The second is post-stall capability (PST) in close air combat which allows the aircraft to perform "controlled tactical maneuvers beyond the maximum lift angle of attack up to at least 70 degrees" [3]. The design of fighter aircraft with

these capabilities requires an understanding of translational and rotational mode interaction as well as nonlinear aerodynamics at large angles of attack and sideslip. Some of these problems of mode interaction and nonlinear dynamics have been observed before the advent of the CCV approach. The phenomena of coupled yaw and pitch divergence at large roll rates was investigated in [4-6]. Mode coupling can also occur at special values of design parameters. For example, the stability of the lateral modes of a rigid aircraft is influenced by the choice of the wing dihedral angle ($Cl\beta$) and the vertical tail size ($Cn\beta$). In Figure D.1 the stability boundaries are plotted in terms of $Cl\beta$ and $Cn\beta$. At $Cl\beta \approx -0.002$ and $Cn\beta \approx 0.0025$ the stability boundaries intersect and the aircraft experiences a simultaneous loss of both Dutch roll and spiral stability. This phenomenon is most likely to occur at high lift coefficients (i.e. large angles of attack).

While the conventional linear model with its assumptions of small angles of attack and sideslip is adequate for the determination of stability boundaries, at large angles of attack and sideslip the aerodynamic derivatives are no longer constant and hence a nonlinear analysis is required. Rhoads and Schuler [7] were one of the first to perform a theoretical and experimental study of airplane dynamics in large-disturbance maneuvers. A key feature of their work is the dependence of the aerodynamic stability derivatives on the Mach number and angle of attack. Unfortunately any possible effects of large sideslip angle were omitted. Since then, NASA has conducted a series of wind tunnel investigation of the effects of large sideslip angle on both static and dynamic stability derivatives (NASA TN 5361, 6091, 6425, 6909, 7972). A sample of these results is reproduced as Figure D.2. The nonlinear behavior of the stability derivatives is evident. It was also observed that the rate or "delay" derivatives due to the sideslip

angle, traditionally omitted from the linear model, may be important for supersonic fighters and their omission can lead to large errors especially in system identification [8]. It is interesting to note that the influence of the rate derivatives was also observed by Tobak and Schiff [9] and Orlik-Ruchemann [10].

Having established that at large disturbances, stability derivatives depend on both the angle of attack and sideslip in a nonlinear manner, the next challenge was to relate these nonlinearities to the dynamics observed at large angles of attack and sideslip. One of the earliest study was carried out by Ross [11] on the HP-115 at the Royal Aircraft Establishment (RAE). Based on a cubic dependence of the yawing moment on the sideslip angle, it was demonstrated that an unstable Dutch roll mode gave rise to a limit cycle. This was perceived during flight as an oscillation of the aircraft wing about the roll axis and is commonly referred to as "wing-rock". This work was followed by investigations of cubic nonlinearities in the rolling moment as well as in the damping-in-roll derivatives [12]. The Gnat trainer was used for this study. In this case both directional divergence as well as wing rock was accounted for. It was also noted that the influence of external stores was significant, confirming the sensitive dependence of the nonlinear dynamics on the aircraft configuration.

Another nonlinear motion was identified by Johnstone and Hogge [13]. In their study of the A-7, they identified certain combinations of angle of attack and sideslip which led to a mutual cancellation of the rolling moments due to the angle of attack and sideslip. The longitudinal and lateral mode coupling resulted in a phenomena called "nose-slice". Basically "nose-slice" refers to a predominantly yawing motion followed by a rapid roll. The nose-slice departure occurred at angles of attack considerably

lower than that for normal stall. This departure was not predicted by the parameter (Cnβ)dyn (NASA TN 6993) commonly used as a measure of spin resistance. This parameter is actually an approximation for the coefficient of the quadratic term in the fourth order characteristic equation for the lateral dynamics. Mathematically the criterion is neither necessary nor sufficient for stability although in practice predictions based on this parameter correlated well with flight test results with few exceptions.

Various methods are available for the analysis of nonlinear systems. Instead of decoupling longitudinal and lateral modes, some researchers have tried to retain as much as possible of the full 6 degree of freedom model. One such method is the pseudo-steady state (PSS) analysis of Young, Schy and Johnson [14,15]. Observing that the effects of gravity are typically small compared to the airspeed for supersonic fighters, a 5th order model was derived. Equilibrium solutions of such a system were referred to as "pseudo-steady". This approach was carried to its extreme by Carroll and Mehra [16], Hui and Tobak [17]. The thrust of their research is the use of bifurcation theory to compute the equilibrium solutions of the full 6 degree of freedom system with nonlinearities based on the interpolation of wind tunnel test results. In the context of bifurcation theory, the Dutch roll/wing rock instability observed by Ross corresponds to a supercritical Hopf bifurcation. This phenomena is characterized by a pair of complex eigenvalues crossing the imaginary axes. The loss of spiral stability is characterized by the crossing of a real eigenvalue. This is referred to as a simple bifurcation. The nonlinear dynamics of aircraft with eigenvalues close to the imaginary axes (i.e. marginally stable /unstable or critical modes) was studied by Cochran and Ho [18] using Malkin's method. Basically Malkin's method is related to the theory of center manifold which

is a dimension reduction technique whereby the critical modes are decoupled from the stable modes and the analysis is then performed on a subsystem of lower dimension.

The extension of these techniques for the analysis of nonlinear systems perturbed by random excitation has not received much attention from the flight dynamics community. Physically this corresponds to flight at large angles of attack and sideslip in a turbulent atmosphere. In the presence of random excitation, the dependence of the stable modes on the critical modes was studied by Haken [19] and is referred to as the "slaving principle". For a stability analysis it is more relevant to consider the dependence of the critical modes on the stable modes. One such method is the extended stochastic averaging theorem of Papanicoloau and Kohler [20] which provides a Markov approximation for the dynamics of the critical modes. Another approach developed by Coullet et al. [21] uses the idea of "normal forms". Once again the emphasis is on dimension reduction and simplification of the resulting subsystem. It was found that for systems perturbed by random excitation, certain nonlinear terms which are removable from the deterministic normal form must be retained due to a phenomena called "stochastic resonance". The methods of stochastic averaging and stochastic normal forms were reconciled by Namachchivaya and Leng [22]. The key result is that the stable modes generated a second order contribution to the critical modes. This was omitted by Coullet et al. [21]. The resulting subsystem was then found to have the same Markov approximation as that given by the extended stochastic averaging theorem.

The problem of mode interaction is not limited to flight dynamics. In aeroelasticity fairly similar phenomena can be observed. The nonlinear oscillation of panel flutter was studied by Dowell [23]. In Figure D.3, the

stability boundaries of a plate in a gas flow are plotted in terms of the nondimensional pressure difference across the plate (λ) and the in-plane loading (R_x/π^2). At $\lambda \approx 200$ and $R_x/\pi^2 \approx -4$, the plate undergoes a simultaneous loss of flutter-divergence stability. Basically flutter is a dynamic instability characterized by a complex pair of eigenvalues crossing the imaginary axis. Hence it may be regarded as an aeroelastic analog of the Dutch roll/wing-rock instability. Similarly, divergence is a static instability and is the counterpart of the spiral instability in flight dynamics. The phenomena of coupled flutter-divergence instability was also observed by Landsberger and Dugundji [24] and Chen and Dugundji [25] in their experimental investigation of the aeroelastic behavior of forward swept graphite/epoxy wings. In Figure D.4, the instability can be seen to occur at special combinations of airspeed and ply angles of the composite fibers. Henceforth for clarity, this aeroelastic perspective will not be emphasized.

Application: Aircraft lateral dynamics at large angles of attack and sideslip in a turbulent atmosphere

The results derived in the previous sections are now applied to the analysis of the lateral dynamics of a rigid aircraft at large angles of attack and sideslip in a turbulent atmosphere. The example is based on the uncontrolled lateral dynamics of the F-104. At a high lift coefficient (i.e. large angle of attack) of $C_L = 0.735$, the nonlinear system is defined by:

$$\frac{d}{dt}\begin{Bmatrix}\beta\\ p\\ r\\ \phi\end{Bmatrix} = \begin{bmatrix} -0.1662 & 0 & -0.9933 & 0.1044\\ -23.99 & -1.390 & 1.292 & 0\\ 4.0772 & -0.0406 & -0.2175 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}\beta\\ p\\ r\\ \phi\end{Bmatrix} + \begin{Bmatrix}0\\ (l\beta 3)\beta^3\\ (n\beta 3)\beta^3\\ 0\end{Bmatrix}$$

$$+ \begin{Bmatrix}0\\ -1.390\\ -0.0406\\ 0\end{Bmatrix} P_g(t)$$

At this lift coefficient the eigenvalues are $0.0532 \pm 2.207j$, 0.000566, -1.88. Hence both the Dutch roll mode and the spiral mode are marginally unstable and the post-critical behavior requires a nonlinear analysis. Following Ross [12], it is assumed that only the roll (p) and yaw (r) equations exhibit significant cubic dependence on the sideslip angle β. For simplicity, only the effects of atmospheric turbulence on the roll rate will be considered. Physically this corresponds to a spanwise velocity variation along the wing causing a rotary motion. The power spectral density (psd) for $P_g(t)$ is given by:

$$\Phi(\omega) = \frac{0.002\,046}{1 + 0.0082\omega^2}$$

where the intensity is taken to be 21 ft/s and the scale factor is 2500 ft for conditions in a thunderstorm [26].

The system is first brought to canonical form using the eigenvectors of the linear system. The transformation is defined by:

$$\begin{Bmatrix} \beta \\ p \\ r \\ \phi \end{Bmatrix} = \begin{bmatrix} -0.1975 & 0.1070 & 0.0146 & 0.1028 \\ 1.889 & 0.7779 & 0.0015 & 5.3462 \\ -0.2287 & -0.3715 & 0.2733 & -0.1215 \\ -0.3317 & 0.8640 & 2.6242 & -2.8436 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ x_{s1} \end{Bmatrix}$$

where x, y represent the Dutch roll (dynamic) mode, z represents the spiral (static) mode and x_{s1}, the stable variable. Converting to polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$) and applying the extended stochastic averaging theorem, the stable mode x_{s1}, may be removed and the Dutch-roll (amplitude), r and the spiral, z, critical modes are denoted by the Ito equations:

$$dr = (\mu r + erz^2 + cr^3 + (S_1)^2/2r)dt + S_1\, dW_1$$

$$dz = (\lambda z + dr^2 z + bz^3)\, dt + S_2\, dW_2$$

where the cubic coefficients of the critical sub-system are now linear functions of $l\beta 3$ and $n\beta 3$:

$$c = -(3.384\, l\beta 3 + 11.90\, n\beta 3)10^{-4} \qquad e = -(5.746\, l\beta 3 + 20.20\, n\beta 3)10^{-6}$$

$$b = (5.324\, l\beta 3 + 31.33\, n\beta 3)10^{-7} \qquad d = (1.88\, l\beta 3 + 11.07\, n\beta 3)10^{-4}$$

In canonical form the external excitation is given by:

$$\begin{Bmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \\ d_s(t) \end{Bmatrix} = \begin{Bmatrix} -0.1122 \\ 0.0492 \\ -0.2767 \\ -0.2273 \end{Bmatrix} P_g(t)$$

and hence using results derived the excitation intensities for the system are:

$$S_1^2 = 0.0075 \int_{-\infty}^{\infty} R_{pp}(\tau)\cos(2.207\tau)\,d\tau$$

$$= 0.0075\,[\,2\pi .\,\Phi(2.207)\,] \qquad \text{(Wiener-Kinchine relation)}$$

$$= 9.277\text{x}10^{-5}$$

$$S_2^2 = 0.07657 \int_{-\infty}^{\infty} R_{pp}(\tau)\,d\tau$$

$$= 0.07657.\,2\pi .\,\Phi(0)$$

$$= 9.845\text{x}10^{-4}$$

For a soft loss of stability it is necessary that b, c < 0, i.e. :

$$l\beta 3 < -5.88\,n\beta 3 \qquad (b < 0)$$

$$l\beta 3 > -3.52\,n\beta 3 \qquad (c < 0)$$

The bifurcation behavior of the deterministic system is preserved if $\delta = 0$, i.e. :

$$\frac{d}{(S_2)^2} = \frac{e}{(S_1)^2}$$

This leads to:

$$l\beta 3 = -5.3\,n\beta 3$$

These conditions are plotted in terms of $n\beta 3$ and $l\beta 3$ in Figure D.5. It can be observed that for the requirements for a potential flow ($\delta = 0$) is well

within the constraints for a soft transition. Furthermore for the values of $n\beta3$ and $l\beta3$ concerned, $\kappa = (d\,(S_1)^2 + e\,(S_2)^2)/4$ is negative. Hence the existence of a normalizable potential flow steady-state pdf for $\delta = 0$ is guaranteed by the conditions $b < 0$ and $c < 0$. Given the actual nonlinear aerodynamic coefficients $n\beta3$ and $l\beta3$, these relations then provide an indication of the dynamics at large angles of attack and sideslip. Since the Dutch roll mode corresponds to a roll-yaw motion, it is preferable in practice to stabilize the Dutch roll mode at the expense of the spiral mode. From the correction factor for the effective unfolding parameters, this is achieved if $\delta > 0$, i.e the nonlinear aerodynamic coefficients should satisfy:

$$l\beta3 \quad > \quad -5.3\, n\beta3$$

This example illustrates the dynamic implications of the nonlinear aerodynamics at large angles of attack and sideslip and it emphasizes the inadequacies of a deterministic nonlinear analysis for systems undergoing a coupled static-dynamic instability.

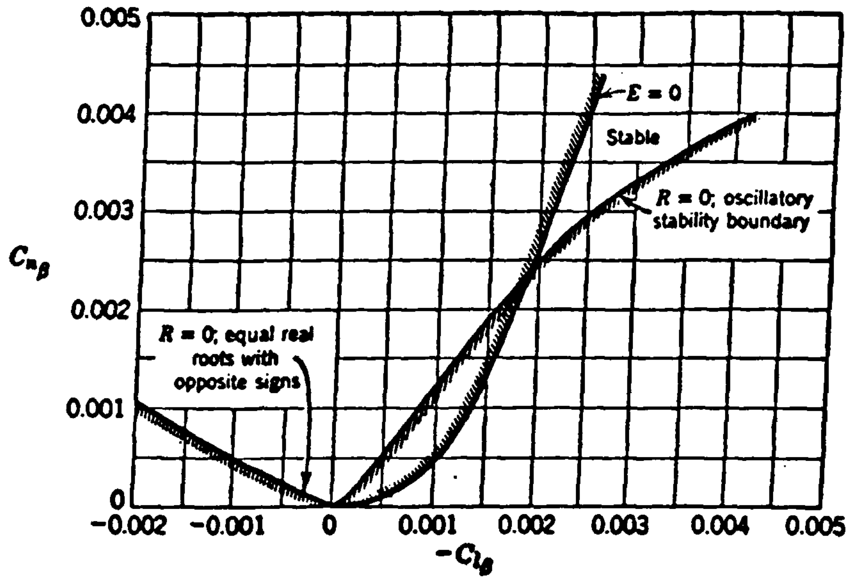

Fig. D.1 : Lateral stability boundaries (NACA report no. 1098)

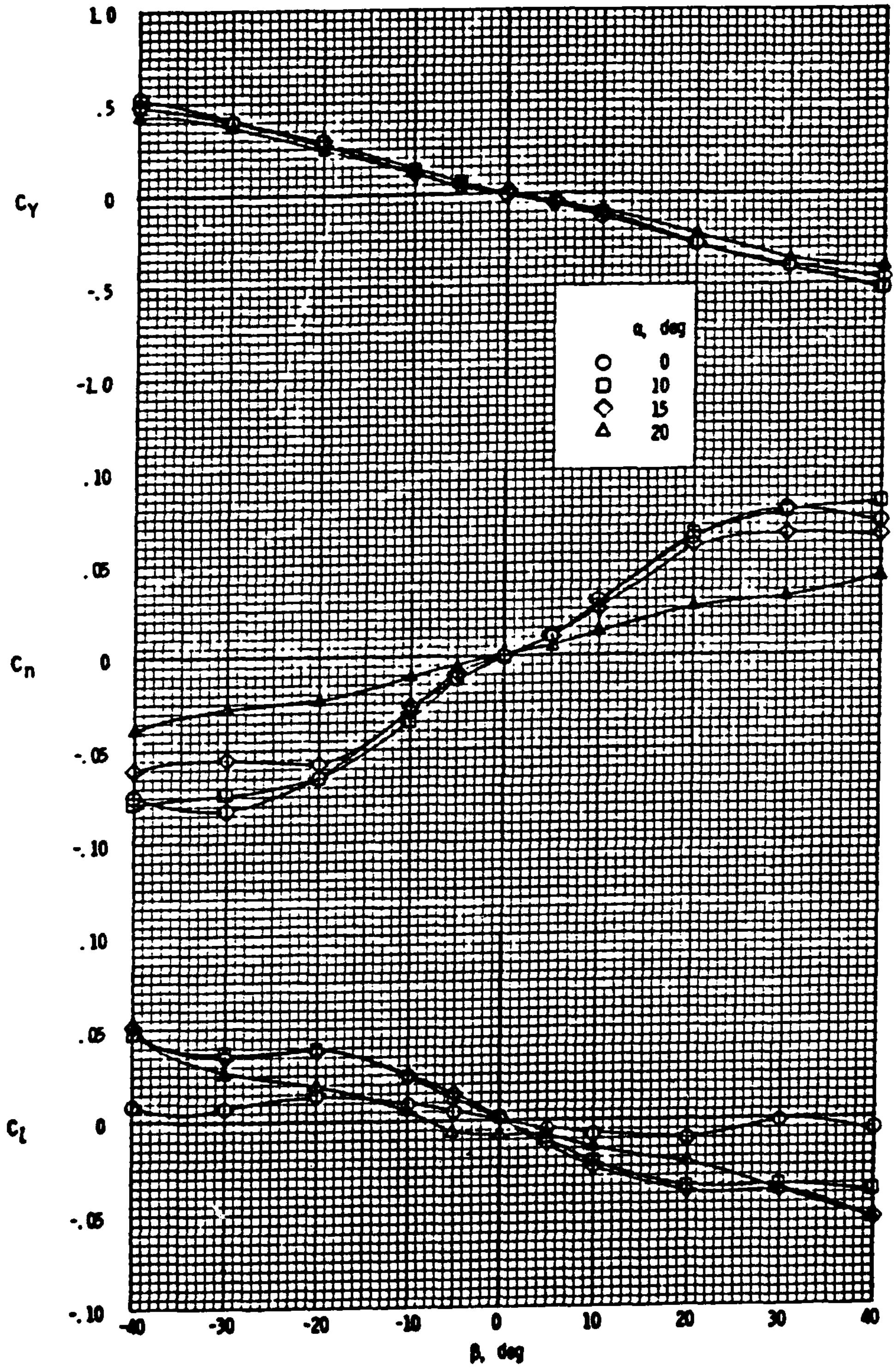

Fig. D.2 : Variation of nondimensional sideforce (C_Y), yawing moment (C_n) and rolling moment (C_l) with angle of attack (α) and sideslip angle (β) (NASA TN 5361)

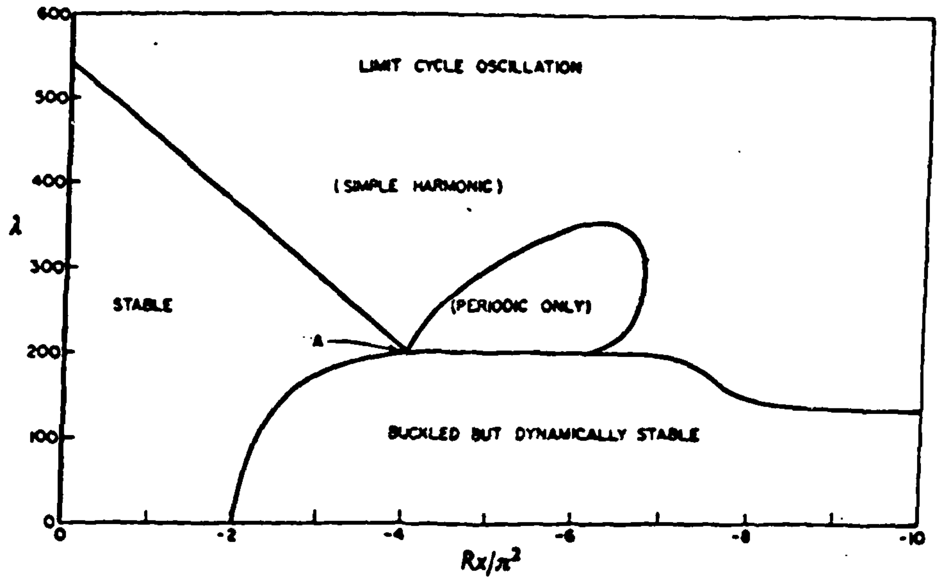

Fig. D.3 : Stability boundaries of an aeroelastic plate

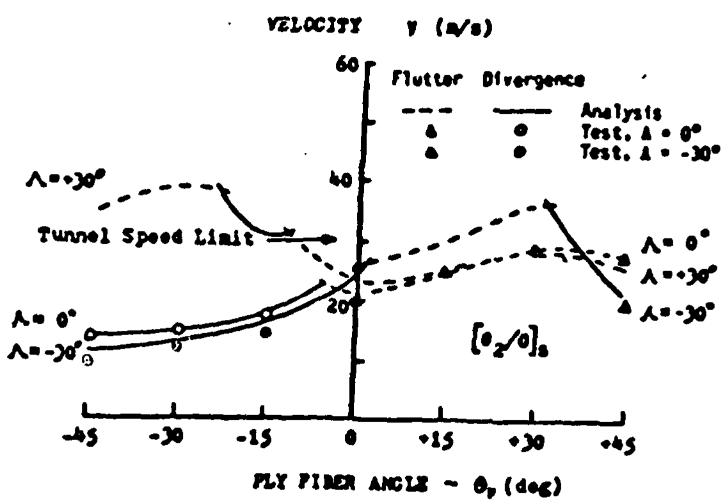

Fig. D.4 : Stability boundaries of a swept-forward composite wing

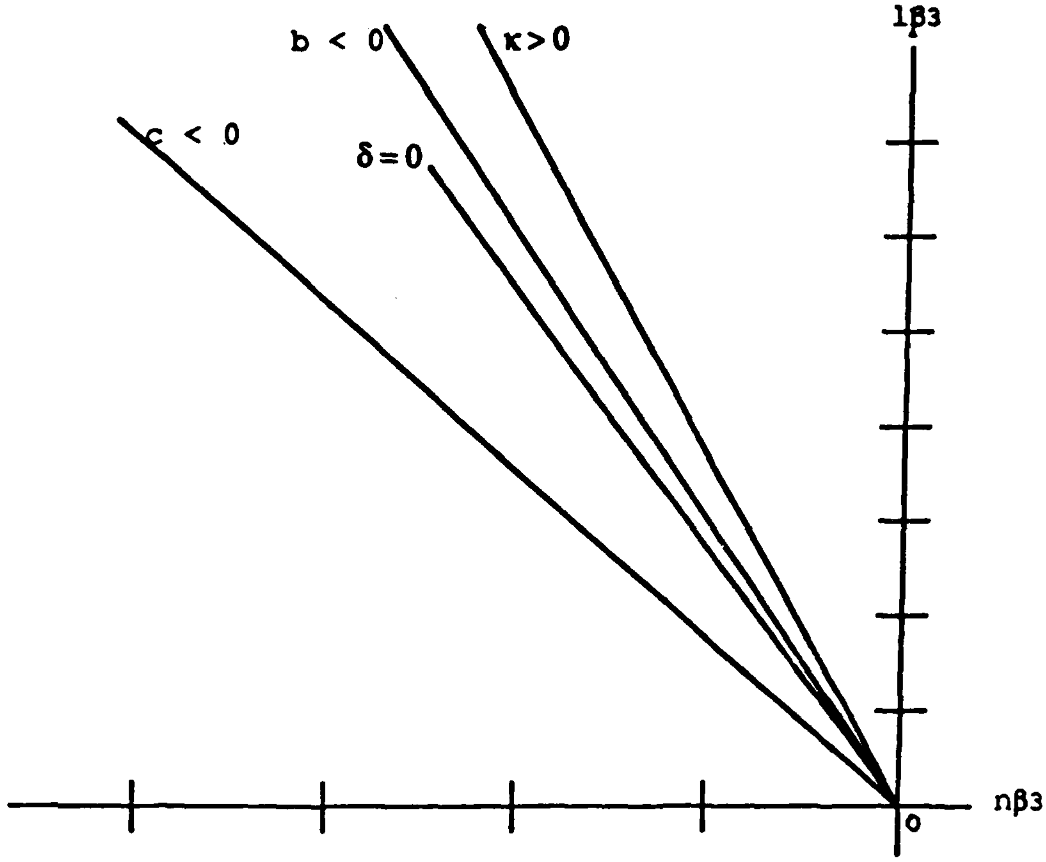

Fig. D.5 : Conditions for a soft loss of stability and potential flow

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APPENDIX E

Introduction

One of the most fundamental components of a mechanical system is a rotating shaft. It is, therefore, not surprising that through the years considerable effort has been directed at obtaining a better understanding of such mechanisms. Toward this end, many problems have been solved. Equally important, many other problems have been better defined (see Dimentberg [1], Biezeno and Grammel [2]). Gyroscopic systems in general, and in particular the problem of rotating shaft received much attention because of its somewhat unexpected results reported by Ziegler [3], in which he showed that the damping tends to destabilize the whirling motion of a shaft-disc system, for angular velocities above the critical angular velocity of the system. The analysis of the system with harmonic parametric perturbations was made by Mettler [4] and Bolotin [5], and various stability boundaries were obtained. Tondl [6] studied the instabilities of a central disc on an asymmetric shaft and obtained equations of an asymmetrical shaft rotating in asymmetrical bearings was made by Gladwell and Stammers [7] using Floquet theory. However, these analyses did not consider the dynamic behavior of the system when parametric excitations are stochastic. The stability of random parametric vibration of shafts have been analyzed by Tam [8], and Schweiger [9] to determine various regions of stochastic instability. In many practical situations, where a shaft may be mounted to other mechanisms, the disturbance arise from both deterministic and random sources. Thus, in this paper we shall examine the response and stochastic stability of rotating shafts when they are excited by random parametric excitations in addition to harmonic parametric excitations. A paper dealing with moment stability of coupled conservative systems under combined harmonic and stochastic excitation was presented by Ariaratnam and Tam [10]. Conditions for stability in the first

and second moments of response were derived for a thin, simply-supported, elastic beam subjected to a small intensity dynamic transverse load at mid-span. This paper follows the approach of Sri Namachchivaya and Ariaratnam [11], Sri Namachchivaya [12] and Ariaratnam and Tam [10] to find mean square stability conditions for the response of the rotating shaft.

Formulation of the Problem

A rotating system can be identified as a gyroscopic system only when treated in a *rotating reference frame*, and this approach will be followed in considering the transverse motion of a continuous uniform elastic shaft of asymmetrical cross-section mounted in a rigid bearing and rotating with constant angular velocity Ω about the horizontal centerline (oz) of the bearings. The rotating shaft of length ℓ, mass per unit length m, and flexural rigidities EI_u, EI_v respectively, parallel to directions ou, and ov is loaded by a time dependent axial thrust, say, $P(t) = P_o\ (1 + f(t))$, as shown in Figure 1a.

The transverse motion of the rotating shaft is given by the following set of two parallel differential equations: (e.g., Dimentberg [1])

$$
\begin{aligned}
&EI_u \frac{\partial^4 u}{\partial z^4} + P(t) \frac{\partial^2 u}{\partial z^2} + m \frac{\partial^2 u}{\partial t^2} + D \frac{\partial u}{\partial t} - 2m\Omega v - m\Omega^2 u = 0 \\
&EI_v \frac{\partial^4 v}{\partial z^2} + P(t) \frac{\partial^2 u}{\partial z^2} + m \frac{\partial^2 v}{\partial t^2} + D \frac{\partial v}{\partial t} + 2m\Omega u - m\Omega^2 v = 0 \, .
\end{aligned}
\tag{1}
$$

For the case of simply supported ends, the following boundary conditions must be satisfied:

$$u(0,t) = u(\ell,t) = 0 \ , \ v(0,t) = v(\ell,t) = 0 \ ,$$

$$\frac{\partial^2 u(0,t)}{\partial z^2} = \frac{\partial^2 v(0,t)}{\partial z^2} = 0 \ , \ \frac{\partial^2 u(\ell,t)}{\partial z^2} = \frac{\partial^2 v(\ell,t)}{\partial z^2} = 0 \ .$$

Considering the fundamental mode, these boundary conditions are satisfied with the following expressions for u and v:

$$u(x,t) = U(t) \sin \pi \frac{z}{\ell}, \quad v(z,t) = V(t) \sin \pi \frac{z}{\ell}$$

We assume that the axial thrust is harmonically varying and is given by

$$P(t) = P_o\left(1 + F(t)\right)$$

where P_o is the mean value, and F(t) consists of a combination of a harmonic term and a stationary stochastic process as perturbations, and assumed to be of the order ε. Substituting the expressions for the modes into the governing partial differential equation and simplifying leads to the following two ordinary differential equations for U and V:

$$\begin{aligned} \ddot{U} - 2\Omega\dot{V} + (\bar{\omega}_1^2 - \Omega^2)\, U + \varepsilon[\zeta U - F(t)\, U] &= 0 \\ \ddot{V} + 2\Omega\dot{U} + (\bar{\omega}_2^2 - \Omega^2)\, V + \varepsilon[\zeta V - F(t)\, V] &= 0 \end{aligned} \qquad (2)$$

where

$$\bar{\omega}_1^2 = \frac{\pi^2}{m\ell^2}\left(P_E^U - P_o\right), \quad \bar{\omega}_2^2 = \frac{\pi^2}{m\ell^2}\left(P_E^V - P_o\right), \quad F(t) = \varepsilon h \cos \nu t + f(t)$$

$$P_E^U = \frac{\pi^2 EI_U}{\ell^2}, \quad P_E^V = \frac{\pi^2 EI_V}{\ell^2}, \quad \zeta = \frac{D}{m}, \quad h = \frac{\pi^2}{m\ell^2},$$

and $\bar{\omega}_1$, $\bar{\omega}_2 (\bar{\omega}_1 < \bar{\omega}_2)$ are the natural frequencies of transverse vibration. Putting $q_1 = U$ and $q_2 = V$, the Lagrange function corresponding to equation (2) with $\varepsilon = 0$, can be written as

$$L(q,\dot{q}) = \frac{1}{2}\left[\dot{q}_1^2 + \dot{q}_2^2 + 2\Omega(q_1\dot{q}_2 - \dot{q}_1 q_2) - (\bar{\omega}_1^2 - \Omega^2)q_1^2 - (\bar{\omega}_2^2 - \Omega)q_2^2\right]$$

Now making use of the relationship $\dot{q}_1 = p_1 + \Omega q_2$, $\dot{q}_2 = p_2 - \Omega q_1$, where $\underset{\sim}{p}$ is the momentum vector conjugate to $\underset{\sim}{q}$, one can write the Hamiltonian as

$$H(\underset{\sim}{q},\underset{\sim}{p}) = \frac{1}{2}\{\underset{\sim}{q}^T,\underset{\sim}{p}^T\}\ S\ \{\underset{\sim}{q},\underset{\sim}{p}\}, \qquad S = \begin{vmatrix} \bar{\omega}_1^2 & 0 & 0 & -\Omega \\ 0 & \bar{\omega}_2^2 & \Omega & 0 \\ 0 & \Omega & 1 & 0 \\ -\Omega & 0 & 0 & 1 \end{vmatrix} = S^T$$

In the above equation, if $\bar{\omega}_1^2 < \Omega^2 < \bar{\omega}_2^2$, S is positive definite and the unperturbed system is stable and the eigenvalues of the system with $\varepsilon = 0$ can be obtained from the equation $|JS - \rho\ I| = 0$. The eigenvalues are distinct and imaginary and are given as

$$\rho_r = \pm\ i\omega_r = \pm i\{(\bar{\omega}_1^2 + \bar{\omega}_2^2 + 2\Omega^2) \pm [(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2 + 8\Omega^2(\bar{\omega}_1^2 + \bar{\omega}_2^2)]^{1/2}\}^{1/2} /\sqrt{2}$$

The Hamilton equation of motion can be written as

$$\dot{\underset{\sim}{Z}} = JS\underset{\sim}{Z} - \varepsilon\ \{\zeta B_1 + B_2 F(t)\}\underset{\sim}{Z}\ , \text{ where } \underset{\sim}{Z} = (\underset{\sim}{q},\underset{\sim}{p}),\ J = \begin{vmatrix} 0 & -I \\ I & 0 \end{vmatrix}. \tag{3}$$

Now consider a canonical transformation $\underset{\sim}{Z} = T\underset{\sim}{y}$ where $\underset{\sim}{y} = (\underset{\sim}{Q},\underset{\sim}{P})$ and

$$T = \begin{bmatrix} a_1 & -a_2\gamma_2 & -a_1 & -a_2\gamma_2 \\ a_1\gamma_1 & a_2 & a_1\gamma_1 & -a_2 \\ a_1\beta_1 & -a_2\alpha_2 & a_1\beta_1 & a_2\beta_2 \\ a_1\alpha_1 & a_2\beta_2 & -a_1\alpha_1 & a_2\beta_2 \end{bmatrix}$$

$$\gamma_1 = \frac{\omega_1^2 - (\bar{\omega}_1^2 - \Omega^2)}{2\Omega\omega_1}, \qquad \gamma_2 = \frac{\omega_2^2 - (\bar{\omega}_2^2 - \Omega^2)}{2\Omega\omega_2}$$

$$\beta_1 = \omega_1 - \Omega\gamma_1 \ , \quad \beta_2 = \omega_2 - \Omega\gamma_2$$

$$\alpha_1 = \Omega - \omega_1\gamma_1 \ , \quad \alpha_2 = \Omega - \omega_2\gamma_2$$

$$a_i = \left(2(\beta_i - \alpha_i\gamma_i)\right)^{1/2}$$

Since the transformation is canonical, the matrix T satisfies the sympletic condition, and the inverse of T can be obtained as

$$T^{-1} = - JT^TJ \ .$$

In addition,

$$T^{-1}ST = J\Omega \ ,$$

where $\Omega = \text{dig}\ \{\omega_1,\omega_2;\omega_1,\omega_2\}$. Thus, Hamilton's equation in the new coordinates $(\underset{\sim}{Q},\underset{\sim}{P})$ can be written as

$$\underset{\sim}{Q} - \Omega\ \underset{\sim}{P} = - \varepsilon\ \{\rho(A_1^{11}\underset{\sim}{Q} + A_1^{12}\underset{\sim}{P}) + \varepsilon^{1/2}F(t)(A_2^{11}\underset{\sim}{Q} + A_2^{12}\underset{\sim}{P})\} \ ,$$

$$\tag{4}$$

$$\underset{\sim}{P} + \Omega\ \underset{\sim}{Q} = - \varepsilon\{\rho(A_1^{21}\underset{\sim}{Q} + A_1^{22}\underset{\sim}{P}) + \varepsilon^{1/2}F(t)(A_2^{21}\underset{\sim}{Q} + A_2^{22}\underset{\sim}{P})\} \ ,$$

where

$$A_1^{11} = \begin{bmatrix} a_1^2(1-\gamma_1^2) & -a_1a_2(\gamma_1+\gamma_2) \\ a_1a_2(\gamma_1+\gamma_2) & a_2^2(1-\gamma_2^2) \end{bmatrix}, \quad A_1^{12} = \begin{bmatrix} -a_1^2(1+\gamma_1^2) & a_1a_2(\gamma_1-\gamma_2) \\ a_1a_2(\gamma_1-\gamma_2) & -a_2^2(1+\gamma_2^2) \end{bmatrix}$$

$$A_1^{21} = \begin{bmatrix} a_1^2(1+\gamma_2^2) & a_1a_2(\gamma_1-\gamma_2) \\ a_1a_2(\gamma_1-\gamma_2) & a_2^2(1+\gamma_2^2) \end{bmatrix}, \quad A_1^{22} = \begin{bmatrix} -a_1^2(1-\gamma_1^2) & -a_1a_2(\gamma_1+\gamma_2) \\ a_1a_2(\gamma_1+\gamma^2) & -a_2^2(1-\gamma_2^2) \end{bmatrix}$$

$$A_2^{11} = \begin{bmatrix} a_1^2\omega_1(1+\gamma_1^2)\ , & a_1a_2[g(1-\gamma_1\gamma_2) - (\alpha_2 + \gamma_1\beta_2)] \\ -a_1a_2[g(1-\gamma_1\gamma_2) - (\alpha_1+\gamma_2\beta_1)]\ , & a_2^2\omega_2(1+\gamma_2^2\ ) \end{bmatrix}$$

$$A_2^{12} = \begin{bmatrix} a_1^2(\beta_1+\gamma_1\alpha_1) \ , & -\ a_1a_2[g(1+\gamma_1\gamma_2) - (\alpha_2-\gamma_1\beta_2)] \\ a_1a_2[g(1+\gamma_1\gamma_2) - (\alpha_1-\gamma_2\beta_1)] \ , & a_2^2[\beta_2+\gamma_2\alpha_2] \end{bmatrix}$$

$$A_1^{21} = \begin{bmatrix} a_1^2(\beta_1+\gamma_1\alpha_1), & a_1a_2[g(1+\gamma_1\gamma_2) - (\alpha_2-\gamma_1\beta_2)] \\ -a_1a_2[g(1+\gamma_1\gamma_2) - (\alpha_1-\gamma_2\beta_1)] \ , & a_2^2(\beta_2+\gamma_2\alpha_2) \end{bmatrix}$$

$$A_2^{22} = \begin{bmatrix} a_1^2\omega_1(1+\gamma_1^2) \ , & -\ a_1a_2[g(1-\gamma_1\gamma_2) - (\alpha_2+\gamma_1\beta_2)] \\ a_1a_2[g(1-\gamma_1\gamma_2) - (\alpha_1+\gamma_2\beta_1)] \ , & \alpha_2^2\omega_2(1+\gamma_2^2)] \end{bmatrix}$$

Introducing a new time $\tau = \nu t$, detuning parameter λ, and a coordinate transformation

$$Q_r = i(\bar{z}_r e^{-i\kappa_r\tau} - z_r e^{i\kappa_r\tau}), \quad P_r = \bar{z}_r e^{-i\kappa_r\tau} + z_r e^{i\kappa_r\tau} \tag{5}$$

where $\kappa_r = \omega_r/\omega_o$, $\nu = \omega_o(1-\varepsilon\lambda)$, z_r are complex variables with conjugates $\bar{z}_r$, in equation (4) yields

$$\begin{aligned} z_r' = (i)\varepsilon\lambda\kappa_r z_r + \frac{1}{2\omega_o}\eta_j \sum_{j=1}^{2} \Big\{ \sum_{s=1}^{2} [(D_{jrs}^{+} + iH_{jrs}^{-})\cos(\kappa_s - \kappa_r)\tau \\ + i\,(D_{jrs}^{+} + iH_{jrs}^{-})\sin(\kappa_s - \kappa_r)\tau]z_s \\ + \sum_{s=1}^{2} [(D_{jrs}^{-} + iH_{jrs}^{+})\cos(\kappa_s + \kappa_r)\tau - i(D_{jrs}^{-} + iH_{jrs}^{+})\sin(\kappa_s + \kappa_r)\tau]\bar{z}_s \Big\} \end{aligned} \tag{6}$$

where $D^{\pm}_{jrs} = A^{(22)}_{jrs} \pm A^{(11)}_{jrs}$, $H^{\pm}_{jrs} = A^{(12)}_{jrs} \pm A^{(21)}_{jrs}$, $\eta_1 = \varepsilon^{1/2}\xi(t)$,

$\eta_2 = -\varepsilon\zeta$, $\xi(t) = \varepsilon^{1/2}h\cos\tau + f(\tau/\nu)$ and the corresponding equation for $\bar{z}_r'$ can be obtained by conjugating equation (6).

Approximation to Markov Process

To a first approximation $z_r(t)$ may be replaced by the solutions of 'averaged' equations using the method of 'stochastic averaging'. According to this procedure, the deterministic terms on the right hand sides of equations (6) are averaged in the usual manner, while the stochastic terms are replaced by their averaged mean plus equivalent fluctuational parts: the details of this procedure may be found in [13]. Applying this procedure to equations (6), it is found that the parametric excitations contribute to the averaged equations only when the frequency of the harmonic excitation is in the neighborhood of the value $2\omega_\ell, \omega_m + \omega_\ell$ and $|\omega_m - \omega_\ell|$; $\ell, m = 1,2$. Thus, for the subharmonic case, i.e., $2\kappa_\ell = 1$, the averaged equations take the form

$$dz_r = -\varepsilon[\beta_{rr} - i(\lambda\kappa_r - \alpha_{rr})]z_r dt + \varepsilon^{1/2}\sum_{j=1}^{4}\sigma_{rj}dw_j \ , \quad r = 1,2,;\ r \neq \ell \ .$$

$$dz_\ell = -\varepsilon\{[\beta_{\ell\ell} - i(\lambda\kappa_\ell - \alpha_{\ell\ell})]z_\ell - \frac{ih}{2\omega_o}[a_\ell^2(1-\gamma_\ell^2)]\bar{z}_\ell\}dt + \varepsilon^{1/2}\sum_{j=1}^{4}\sigma_{\ell j}\,dw_j \ , \quad \ell = 1,2;\ \ell \neq r \ . \tag{7}$$

For combination resonance of the type $\omega_o = \omega_1 + \omega_2$, the averaged equations take the form;

$$dz_1 = -\varepsilon\{[\beta_{ii} - i(\lambda\kappa_1 - \alpha_{11})]z_1 - \frac{ih}{2\omega_o}[a_1a_2(\gamma_1 - \gamma_2)]\bar{z}_2\}dt$$

$$+ \varepsilon^{1/2}\sum_{j=1}^{4}\sigma_{1,j}\,dw_j \tag{8}$$

$$d\bar{z}_2 = -\varepsilon\{[\beta_{22} + i(\lambda\kappa_2 - \alpha_{22})]\bar{z}_2 + \frac{ih}{2\omega_o}[a_1a_2(\gamma_1 - \gamma_2)]z_1\}dt$$

$$+ \varepsilon^{1/2}\sum_{j=1}^{4}\sigma_{4,j}\,dw_j$$

Similarly, the averaged equations for $\omega_o = |\omega_1 - \omega_2|$ takes the form

$$dz_1 = -\varepsilon\{[\beta_{11} - i(\lambda\kappa_1 - \alpha_{11})]z_1 + \frac{h}{2\omega_o}[a_1a_2(\gamma_1 + \gamma_2)]z_2\}dt$$

$$= \varepsilon^{1/2}\sum_{j=1}^{4}\sigma_{1j}\,dw_j \tag{9}$$

$$dz_2 = -\varepsilon\{[\beta_{22} - i(\lambda\kappa_2 - \alpha_{22})]z_2 - \frac{h}{2\omega_o}[a_1a_2(\gamma_1 + \gamma_2)]z_1\}dt$$

$$+ \varepsilon^{1/2}\sum_{j=1}^{4}\sigma_{2j}\,dw_j$$

In the above equations

$$\beta_{rr} = \zeta\kappa_r\frac{\Omega(\gamma_r + 1/\gamma_r)}{(\omega_r^2 - \omega_s^2)} + \frac{\Delta^2}{2\omega_o^2}\{(\gamma_r + 1/\gamma_r)^2 S_{ff}(0) - (\gamma_r - 1/\gamma_r)^2 S_{ff}(2\kappa_r)$$

$$- \frac{(\gamma_r + \gamma_s)^2}{\gamma_r\gamma_s}S_{ff}(\kappa_r - \kappa_s) + \frac{(\gamma_r - \gamma_s)^2}{\gamma_r\gamma_s}S_{ff}(\kappa_r + \kappa_s)\} \tag{10a}$$

$$\alpha_{rr} = \frac{\Delta^2}{2\omega_o^2}\{(\gamma_r - 1/\gamma_r)^2\psi_{ff}(2\kappa_r) + \frac{(\gamma_r + \gamma_s)^2}{\gamma_r\gamma_s}\psi_{ff}(\kappa_r - \kappa_s)$$

$$- \frac{(\gamma_r - \gamma_s)^2}{\gamma_r\gamma_s}\psi_{ff}(\kappa_r + \kappa_s)\}$$

$$[\sigma\sigma^T]_{r,s} = -(-1)^{r+s}\frac{\Delta^2}{\omega_o^2} z_r z_s\{(\gamma_r + 1/\gamma_r)(\gamma_s + 1/\gamma_s)S_{ff}(0)$$

$$+ (1-\delta_{rs})\frac{(\gamma_r + \gamma_s)^2}{\gamma_r\gamma_s} S_{ff}(k_r - k_s)\} \qquad r,s = 1,2.$$

$$[\sigma\sigma^T]_{r,2+r} = \frac{\Delta^2}{\omega_o^2}\{z_r\bar{z}_r[(\gamma_r + 1/\gamma_r)^2 S_{ff}(0) + (\gamma_r - 1/\gamma_r)^2 S_{ff}(2\kappa_r)]$$

$$- z_s\bar{z}_s[\frac{(\gamma_r + \gamma_s)^2}{\gamma_r\gamma_s} S_{ff}(k_r - k_s) + \frac{(\gamma_r - \gamma_s)^2}{\gamma_r\gamma_s} S_{ff}(\kappa_r + k_s)]\} \quad r = 1,2;\ s \neq r$$

$$[\sigma\sigma^T]_{r,2+s} = -\frac{\Delta^2}{\omega_o^2}\{z_r\bar{z}_s[(\gamma_r+1/\gamma_r)(\gamma_s+1/\gamma_s) + \frac{(\gamma_r - \gamma_s)^2}{\gamma_r\gamma_s} S_{ff}(k_r+k_s)]\} r,s=1,2;s\neq r$$

$$S_{ff}(\omega) = \int_0^\infty R_{ff}(\tau)\cos\omega\tau d\tau\ ,\ \psi_{ff}(\omega) = \int_0^\infty R_{ff}(\tau)\ \sin\omega\tau d\tau\ ,\ \Delta = \Omega/(\omega_1^2 - \omega_2^2)$$

$W_j(j = 1,2...2n)$, are independent Wiener processes of unit intensity and the remaining terms are as defined in the previous equations.

It is worth pointing out that for the white noise case, i.e. $S(\omega) = S_o$ and $\psi(\omega) = 0$, the above quantities reduce to

$$\beta_{rr} = \left(\frac{\zeta}{2\omega_o}\right) \frac{1}{(\omega_r^{\,2} - \omega_s^{\,2})} \left\{2\omega_r^{\,2} - (\bar{\omega}_r^{\,2} + \bar{\omega}_s^{\,2} - 2\Omega^2)\right\} , \quad \alpha_{rr} = 0$$

$$[\sigma\sigma^T]_{r,s} = -(-1)^{r+s} \frac{\Delta^2}{\omega_o^2} \left(\frac{z_r z_s}{\gamma_r \gamma_s}\right)\left\{(\gamma_r^2 + 1)(\gamma_s^2 + 1) + (1-\delta_{rs})(\gamma_r + \gamma_s)^2\right\} S_o$$

$$r,s = 1,2$$

$$[\sigma\sigma^T]_{r,2+r} = \frac{2\Delta^2}{\omega_o^2} \left\{z_r \bar{z}_r(\gamma_r^2 + 1/\gamma_r^2) - z_s \bar{z}_s(\gamma_r/\gamma_s + \gamma_s/\gamma_r)\right\} S_o \qquad (10b)$$

$$r \neq s, \; r = 1,2$$

$$[\sigma\sigma^T]_{r,2+s} = -\frac{\Delta^2}{\omega_o^2} \left(\frac{z_r \bar{z}_s}{\gamma_r \gamma_s}\right)\left\{(\gamma_r^2 + 1)(\gamma_s^2 + 1) + (\gamma_r - \gamma_s)^2\right\} S_o$$

$$r \neq s, \; r,s = 1,2$$

Stability Analysis

First moments stability is considered in this section. The differential equations governing the first moments are obtained by taking the expectations of both sides of equations (7-9). It is evident that the resulting equations will be the same as equations (7-9) with the stochastic terms absent and the variables z_r, $\bar{z}_r$ replaced by their expectations. Since these equations are linear, the conditions for stability in the first moments can be found readily with the help of the Routh-Hurwitz criterion. In the remainder of this section, the moment stability conditions are obtained first for the subharmonic resonance case $\omega_o = 2\omega_\ell$ and then for the combination type resonances, $\omega_o = |\omega_1 \pm \omega_2|$.

Subharmonic Resonance

Setting $\ell = m$ and $\kappa_m = 1/2$ in equations (7) we obtain the following stability conditions

$$\beta_{rr} > 0 \quad r = 1,2, \; r \neq m \tag{11a}$$

$$(1 - 2\alpha_{mm} - \nu/\omega_o)^2 > \{(\frac{h}{\omega_o^2})^2 (\frac{(\bar{\omega}_1^{-2} - \bar{\omega}_2^{-2})^2}{(\omega_1^2 - \omega_2^2)^2}) - 4\beta_{mm}^2\}, \; m = 1,2, \; m \neq r \tag{11b}$$

It is evident that the results for the case of harmonic excitation (at $\omega_o = 2\omega_m$) can be deduced from inequalities (11) when the stochastic terms are removed. The conditions (11a) depend only on the damping term and the stochastic part of the excitation, whereas the inequality (11b) is a modification due to the stochastic term of the known stability condition derived for the case of harmonic excitation.

Combination resonance : $\omega_o = |\omega_1 \pm \omega_2|$

For combination resonance, we begin the analysis with the case $\kappa_1 + \kappa_2 = 1$. For this case, the stability conditions are found to be

$$\beta_{rr} > 0 \quad r = 1,2 \tag{12a}$$

$$(1 - (\alpha_{11} + \alpha_{22}) - \nu/\omega_o)^2 > \epsilon^2\{[\beta_{11}/\beta_{22}]^{1/2} + [\beta_{22}/\beta_{11}]^{1/2}\}^2$$

$$x\{(\frac{h\Omega}{2\omega_o^2})^2 \frac{[(\bar{\omega}_1^{-2} - \Omega^2)^{1/2} + (\bar{\omega}_2^{-2} - \Omega^2)^{1/2}]^2}{(\omega_1 - \omega_2)^2 \, \omega_1\omega_2} - \beta_{11}\beta_{22}\} \tag{12b}$$

It is evident that for $\beta_{11}\beta_{22} > 0$, we must have $\Omega < \bar{\omega}_1$ for the existence of stability boundaries in the $(h, \nu/\omega_o)$ parameter space. Similarly, the stability conditions for the case $|\kappa_1 - \kappa_2| = 1$ are found to be

$$\beta_{rr} > 0 \quad r = 1,2 \tag{13a}$$

$$(1 - (\alpha_{11} - \alpha_{22}) - \nu/\omega_o)^2 > \epsilon^2\{[\beta_{11}/\beta_{22}]^{1/2} + [\beta_{11}/\beta_{22}]^{1/2}\}^2$$

$$\times\left\{ - \left(\frac{\Omega h^*}{2\omega_o^2}\right)^2 \frac{[(\bar{\omega}_1^2 - \Omega^2)^{1/2} - (\bar{\omega}_2^2 - \Omega^2)^{1/2}]^2}{(\omega_1 + \omega_2)^2\, \omega_1\omega_2} - \beta_{11}\beta_{22}\right\} , \tag{13b}$$

Furthermore, for $\Omega < \bar{\omega}_1$ and $\beta_{rr} > 0$ the system will be always stable.

Again, the conditions (11a), (12a) and (13a) depend only on the damping term and the stochastic part of the excitation, while the conditions (11b), (12b) and (13b) are modifications due to the stochastic term of the known stability condition for the harmonic excitation. In the absence of stochastic excitation or when the stochastic excitation is a white noise, the conditions reduce to the inequalities for the case of harmonic excitation, with

$$(\beta_{11})^{\pm} = \left(\frac{\zeta}{2\omega_o^2}\right) \frac{1}{(\omega_1 \mp \omega_2)} \left[2\omega_1^2 - (\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2)\right] ,$$

$$(\beta_{22})^{\pm} = \left(\frac{\zeta}{2\omega_o^2}\right) \frac{-1}{(\omega_1 \mp \omega_2)} \left[2\omega_2^2 - (\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2)\right] , \tag{14}$$

$$(\beta_{11}\beta_{22})^{\pm} = \left(\frac{\zeta}{2\omega_o^2}\right)^2 \frac{1}{(\omega_1 \mp \omega_2)^2} \left[(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2 + 8\Omega^2\, (\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2)\right]$$

It is obvious that for the undamped system ($\zeta = 0$) the first moments are always stable (critically) when $\omega_o = \omega_1 + \omega_2$ and $\omega_o = |\omega_1 - \omega_2|$ in the regions $\Omega > \bar{\omega}_2$ and $\Omega < \bar{\omega}_1$, respectively (see Figure 1b). Since the natural frequencies are ordered, $\omega_1 > \omega_2$, and the inequality $2\omega_2^2 < \bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2 < 2\omega_1^2$ holds, when the damping is present ($\zeta \neq 0$), for $\Omega^2 < \bar{\omega}_1^2$, we have $(\beta_{11}\beta_{22})^{\pm} > 0$ and $(\beta_{11})^{\pm} > 0$, $(\beta_{22})^{\pm} > 0$ and the stability is governed by the conditions (11b) and (12b) for the cases $2\kappa_1 = 1$ and $\kappa_1 + \kappa_2 = 1$, respectively. Furthermore, for $\Omega^2 > \bar{\omega}_2^2$, we have $(\beta_{11})^{\pm} > 0$, $(\beta_{22})^{\pm} < 0$ and $(\beta_{11}\beta_{22})^{\pm} < 0$, implying that

the first moments are always unstable regardless of conditions (11b) and (12b). This indeed agrees with the results of Chetayev [14], which states that the addition of complete dissipation will destabilize the system which was originally stabilized by gyroscopic forces. Thus, one can conclude that for white noise excitation first moment stability conditions are identical to the stability conditions of harmonic excitation and the addition of damping destabilizes the region $\Omega^2 > \bar{\omega}_2^2$. However, when the excitations are not white noise but a band limited excitation then it is evident from Eq. (10), that by choosing the spectrum of the excitation near 0, $2\omega_r$, $\omega_1 + \omega_2$, $|\omega_1 - \omega_2|$ one may stabilize the system in the region $\Omega^2 > \bar{\omega}_2^2$, and is explained below.

Returning now to the non-white noise excitation, consider some particular forms of the excitation spectral density $S(\omega)$ which vanishes outside the band with $\omega_o - 1/2\,\Delta\omega_o < \omega < \omega_o + 1/2\,\Delta\omega_o$, such that the correlation time is $O(1/\Delta\omega_o)$. Therefore, if $\Delta\omega_o >> \varepsilon$, the Markov approximation by the use of the limit theorem will remain valid. Thus, considering $\omega_o \approx 0$, and $\omega_o \approx |\omega_1 + \omega_2|$ will definitely make β_{11} and β_{22} positive for large values of S_o as is evident from Eq. (10). It may be noted that $\gamma_1\gamma_2 > 0$ for $\Omega^2 > \bar{\omega}_2^2$. Thus, by appropriately choosing the form of the excitation spectrum, an unstable rotating system can be stabilized.

Second Moment Stability

Even though the calculation of the explicit formulas in terms of shaft parameters for the mean square stability is long and cumbersome, the equations governing the second order moments along with their characteristic equation are given below. However, when the periodic excitation is absent, explicit mean square stability conditions can be obtained directly by letting h be identically zero. The differential equations governing the second moments are obtained by taking the expectation of the Itô equations corresponding to the

norm $\{z_r\bar{z}_r\}$, r = 1,2. Thus, applying the Itô differential rule leads to the following equations for the cases $\kappa_1 + \kappa_2 = 1$ and $\kappa_1 - \kappa_2 = 1$.

Case 1: $\kappa_1 + \kappa_2 = 1$

$$\begin{aligned}
\frac{d}{dt}\langle z_1\bar{z}_1\rangle &= 2\epsilon\{[-\beta_{11} + S_{11}]\langle z_1\bar{z}_1\rangle + [S_{12}]\langle z_2\bar{z}_2\rangle \\
&\quad - iH^+[\langle z_1 z_2\rangle - \langle\bar{z}_1\bar{z}_2\rangle]\} \\
\frac{d}{dt}\langle z_2\bar{z}_2\rangle &= 2\epsilon\{[S_{12}]\langle z_1\bar{z}_1\rangle + [-\beta_{22} + S_{22}]\langle z_2\bar{z}_2\rangle \\
&\quad - iH^+[\langle z_1 z_2\rangle - \langle\bar{z}_1\bar{z}_2\rangle]\} \\
\frac{d}{dt}\langle z_1 z_2\rangle &= 2\epsilon\{iH^+[\langle z_1\bar{z}_1\rangle + \langle z_2\bar{z}_2\rangle] \\
&\quad = \frac{1}{2}[-(\beta_{11} + \beta_{22}) - (S^+_{\phi\phi} + 2S_{12}) + i\lambda]\langle z_1 z_2\rangle\}
\end{aligned} \tag{15}$$

Case 2: $\kappa_1 - \kappa_2 = 1$

$$\begin{aligned}
\frac{d}{dt}\langle z_1\bar{z}_1\rangle &= 2\epsilon\{[-\beta_{11} + S_{11}]\langle z_1\bar{z}_1\rangle + [S_{12}]\langle z_2\bar{z}_2\rangle \\
&\quad - H^-[\langle z_1\bar{z}_2\rangle + \langle\bar{z}_1 z_2\rangle]\} \\
\frac{d}{dt}\langle z_2\bar{z}_2\rangle &= 2\epsilon\{[S_{12}]\langle z_1\bar{z}_1\rangle + [-\beta_{22} + S_{22}]\langle z_2\bar{z}_2\rangle \\
&\quad + H^-[\langle z_1\bar{z}_2\rangle + \langle\bar{z}_1 z_2\rangle]\} \\
\frac{d}{dt}\langle z_1\bar{z}_2\rangle &= 2\epsilon[H^-[\langle z_1\bar{z}_1\rangle - \langle z_2\bar{z}_2\rangle] \\
&\quad + \frac{1}{2}[-(\beta_{11} + \beta_{22}) + (S^-_{\phi\phi} + 2S_{12}) + i\lambda]\langle z_1\bar{z}_2\rangle\}
\end{aligned} \tag{16}$$

where

$$S_{11} = \frac{S_o\Delta^2}{\omega_o^2} * \{2 + (\bar{\omega}_1^2 - \bar{\omega}_2^2)^2/4\Omega^2\omega_1^2\},\quad S_{22} = \frac{S_o\Delta^2}{\omega_o^2} * \{2 + (\bar{\omega}_1^2 - \bar{\omega}_2^2)/4\Omega^2\omega_2^2\}$$

$$S_{12} = \frac{S_o\Delta^2}{\omega_o^2}\left\{\frac{\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2}{\omega_1\omega_2}\right\}, \quad H^{\mp} = \frac{h}{4\omega_o}\left[a_1 a_2(\gamma_1 \pm \gamma_2)\right]$$

$$S_{\phi\phi}^{\pm} = \frac{S_o\Delta^2}{\omega_o^2}\left\{\frac{(\omega_1 \pm \omega_2)^2(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2(\Omega^2 + \omega_1\omega_2))}{4\Omega^2\omega_1\omega_2}\right\}$$

and the rest of the terms are defined in eqs. (10b). In deriving eqs. (15) and (16), the stochastic excitation is assumed a white noise.

Seeking solutions proportional to exp $(\bar{\lambda}\tau)$, the characteristic equation for the exponent $\bar{\lambda}$ is obtained as

$$\bar{\lambda}^4 + a_3\bar{\lambda}^3 + a_2\bar{\lambda}^2 + a_1\bar{\lambda} + a_0 = 0$$

where a_i's for both the cases are given in Appendix - A. It may be noted that even though the matrix equations (15) and (16) are complex, the characteristic equation is real and stability conditions can be easily obtained using Routh-Hurwitz criterion. Thus, the mean square stability condition can be written as

$$a_i > 0 \quad \text{and} \quad a_1 a_2 a_3 > a_1^2 - a_3^2 a_0$$

Due to the complexity of the algebra, explicit form of the stability conditions in terms of system parameters are not derived. However, when the excitation is purely stochastic (white noise), the second moment stability conditions can be derived by letting $h = 0$, $\tau = \omega_o t$, and $S_{ff}(\kappa_r) = \omega_o S_{ff}(\omega_r)$ in equations (15) and (16) as

$$(\zeta/\Omega^2)[4\Omega^2 + (\omega_1^2 - \omega_2^2)] > S_o \frac{[8\Omega^2\omega_1^2 + (\bar{\omega}_1^2 - \bar{\omega}_2^2)^2]}{4\Omega^2\omega_1^2(\omega_1^2 - \omega_2^2)} ,$$

$$(\zeta/\Omega^2)[4\Omega^2 + (\omega_2^2 - \omega_1^2)] > S_o \frac{[8\Omega^2\omega_2^2 + (\bar{\omega}_1^2 - \bar{\omega}_2^2)^2]}{4\Omega^2\omega_2^2 (\omega_2^2 - \omega_1^2)} , \tag{17}$$

$$16\zeta^2[(\omega_1^2 - \omega_2^2)^2 - 16\Omega^4] - \frac{2\zeta S_o}{\omega_1^2\omega_2^2(\omega_1{}^2 - \omega_2{}^2)} \{[(\omega_1^2 - \omega_2^2) + 4\Omega^2]$$

$$\cdot [(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2 + 8\Omega^2\omega_2^2]\omega_1^2$$

$$+ [(\omega_2^2 - \omega_1^2) + 4\Omega^2][(\bar{\omega}_1^2 - \omega_1^2)^2 + 8\Omega^2\omega_1^2]\ \omega_2^2\}$$

$$+ [(\bar{\omega}_1^2 - 2\Omega^2) / (\bar{\omega}_2^2 - 2\Omega^2) - (\bar{\omega}_2^2 - 2\Omega^2)/(\bar{\omega}_1{}^2 - 2\Omega^2)]^2 > 0$$

The mean square stability results for the non-white noise excitation is given in Appendix - B.

Special Case: Shaft with Symmetric Cross-Section

The results presented in the previous sections can be reduced further for a shaft with symmetric cross-section and the simplified stability conditions are presented in this section. For a shaft with symmetric cross-section, i.e., $\bar{\omega}_1^2 = \bar{\omega}_2^2 = \bar{\omega}^2$, the natural frequencies of the system reduce to $\omega_1 = \bar{\omega} + \Omega$ and $\omega_2 = \bar{\omega} - \Omega$ when $\bar{\omega} > \Omega$. For this case, the value of $\gamma_1 = -\gamma_2 = 1$. The first and second moment stability conditions are obtained for the case $k_r + k_s = 1$ and are given below.

First moment stability condition

(1) non-white noise excitation

$$(1-2\alpha - \nu/2\bar{\omega})^2 > \{(\bar{\beta}_{11}/\bar{\beta}_{22})^{1/2} + (\bar{\beta}_{22}/\bar{\beta}_{11})^{1/2}\}^2 \cdot (\frac{1}{8\bar{\omega}^2})^2\{h^2 - 4\zeta^2\bar{\beta}_{11}\bar{\beta}_{22}\} \quad (18)$$

where $\alpha = (\frac{1}{32\bar{\omega}^4})\ \psi_{ff}(1)$

$$\bar{\beta}_{ii} = \{\omega_i + (\frac{1}{8\zeta\bar{\omega}^2})\ [S_{ff}(0) - S_{ff}(1)]\}$$

(2) white noise excitation

$$(1-\nu/2\bar{\omega})^2 > \varepsilon^2\{(\frac{\bar{\omega}-\Omega}{\bar{\omega}+\Omega})^{1/2} + (\frac{\bar{\omega}-\Omega}{\bar{\omega}+\Omega})^{1/2}\}^2 \cdot (\frac{1}{8\bar{\omega}^2})^2\{h^2-4\zeta^2(\bar{\omega}^2 - \Omega^2)\} \quad (19)$$

Second moment stability conditions (white noise excitation)

$$16H^2(D + 2S) + \lambda^2[4S - D(1 - (\Omega/\bar{\omega})^2)] + (\Omega/\bar{\omega})^2 D(D + 2S)^2 + D(12S^2 - D^2) + 16S^3 < 0$$

$$a_1 = 16H^2(D + S) + \lambda^2(2S - D) + (\Omega/\bar{\omega})^2 D^2(D + 2S) + D(12S^2 - 2D^2) + 8S^3 < 0$$

$$16H^2 - \lambda^2 + (\Omega/\bar{\omega})^2 D^2 - 6D^2 + 12S^2 < 0$$

$$4a_1[16H^2(S-D) + \lambda^2(D+2S) + (\Omega/\bar{\omega})^2 D^2(2S-D) + 6D^3-12DS^2+8S^3] - D^3P < 0$$

where

$$P = 16H^2S + \lambda^2(2S + (\Omega/\bar{\omega})^2 d) + (\Omega/\bar{\omega})^2 2DS(D+2S) + D^3+8S^3 > 0 ,$$

$$H = \frac{h}{16\bar{\omega}^2} , \quad S = \frac{S_o}{32\bar{\omega}^4} , \quad \text{and} \quad D = \frac{\zeta}{2\bar{\omega}}$$

As before, for the case of purely stochastic excitation, i.e., $h = 0$, the stability conditions reduce to

$$\zeta > \frac{S_{ff}(2\bar{\omega})}{2(\omega^2 - \Omega^2)}$$

for the case $\bar{\omega} > \Omega$. However, for $\bar{\omega} < \Omega$, instability always occurs. Furthermore, for $\Omega = 0$, the above condition reduces to the well-known results of a column under random axial loading.

The numerical results for this special case for various values of H, S, D, $\bar{\omega}$ and Ω are given in Figures 2, 3, 4 and 5. In Figures 2(a) and 2(b), the first and second moment stability conditions are plotted respectively for different values of damping parameters. In Figure 3, the first moment stability regions are compared for the white noise and non-white noise excitations. For the non-white noise case the stability region shifts to the right. The first and second moment stability regions are compared in Figure 4. It can be seen that the second moment stability region is smaller than that of the first moment as one expects. Finally, the mean square stability conditions for purely white noise excitation is given in Figures 5(a) and 5(b). The stability regions are compared for different values of the shaft speed Ω.

Conclusions

An analytical method, based on sympletic transformation and theory of both deterministic and stochastic averaging has been presented for investigating a rotating shaft under combined harmonic and stochastic excitations of small intensity. Since a rotating shaft being one of the most fundamental components of many mechanical systems, such forms of excitations

are realistic ones to assume in many practical situations where the disturbance arise from both deterministic and non-deterministic sources.

The equations of motion were first transformed to first order Hamiltons equation and applying appropriately deterministic and stochastic averaging, the state variables under suitable conditions, converge in a weak sense to Markov vector which satisfies Itô equations. From the Itô equations, conditions for first and second order moments were obtained, with the aid of Routh-Hurwitz criteria. It was shown that the results for harmonic excitation case can be obtained from the first moment stability conditions by making the stochastic terms identically zero. For the white noise excitation, first moment stability conditions are identical to the stability conditions of harmonic excitation. Furthermore, it was observed that stabilization of harmonic parametric instabilities are possible when the excitation is band limited with certain spectrum.

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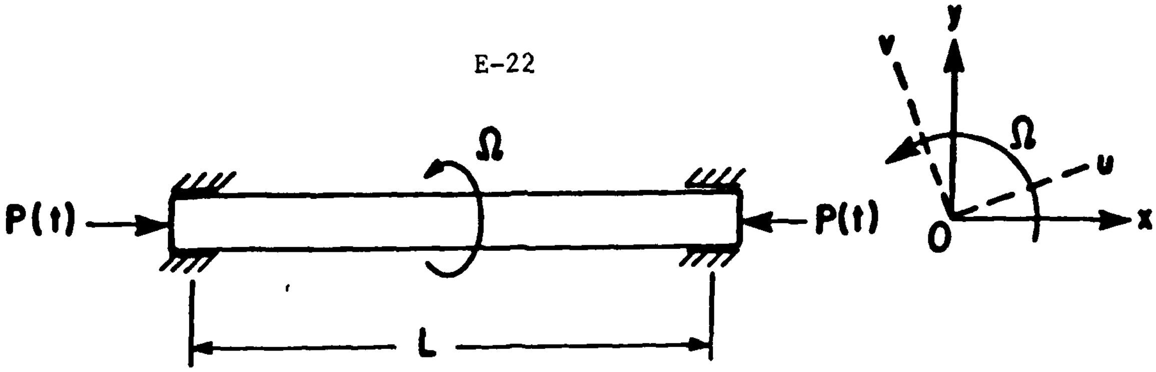
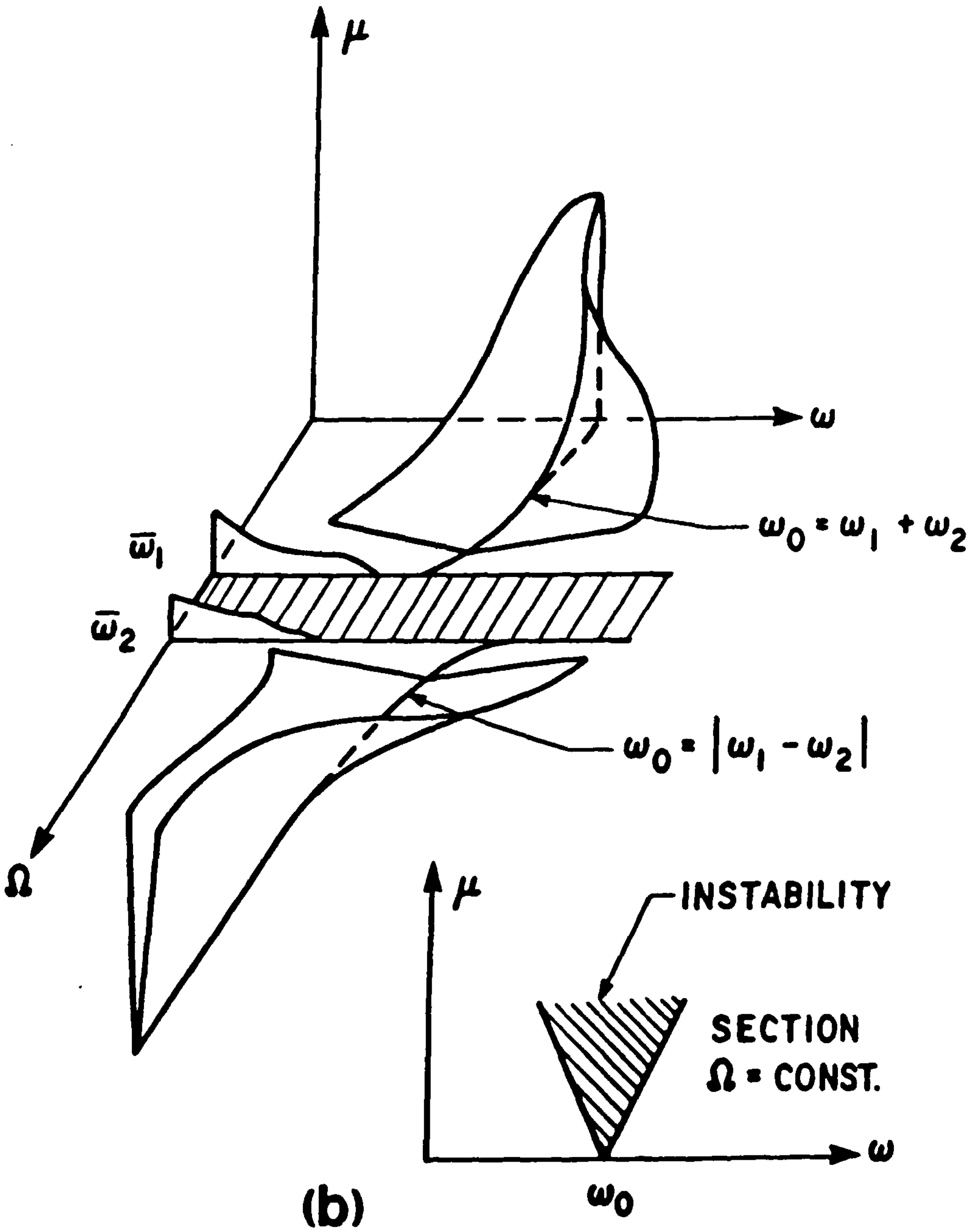
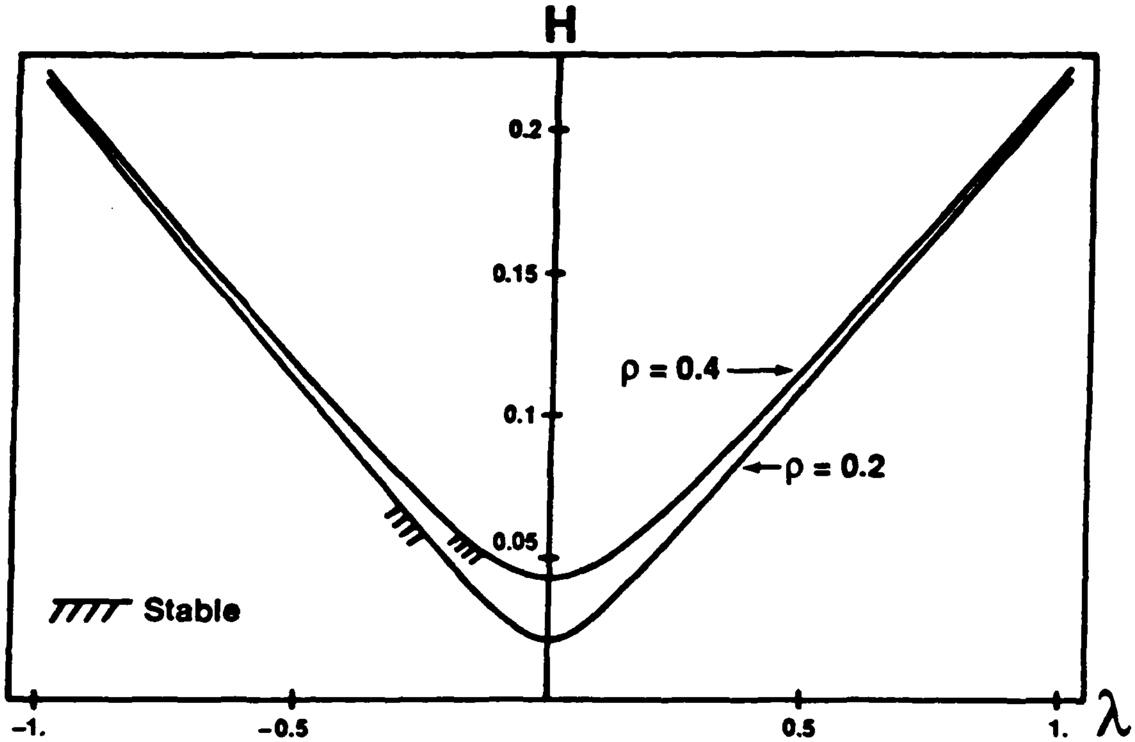

Ω
P(t)
P(t)
L
y
v
Ω
u
O
x

(a)

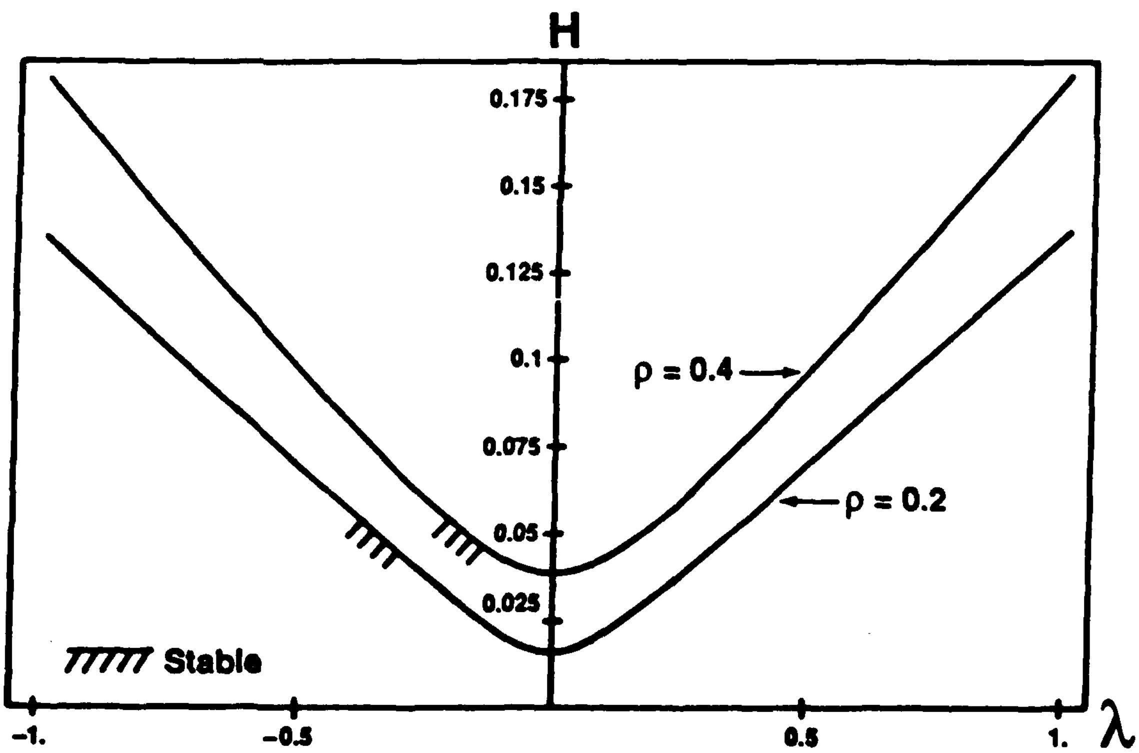

μ
ω
$\bar{\omega}_1$
$\bar{\omega}_2$
$\omega_0 = \omega_1 + \omega_2$
$\omega_0 = |\omega_1 - \omega_2|$
Ω
μ
INSTABILITY
SECTION
Ω = CONST.
ω
ω_0

(b)

$S = 0.01,\ \bar{\omega} = 1,\ \Omega = 0.5$

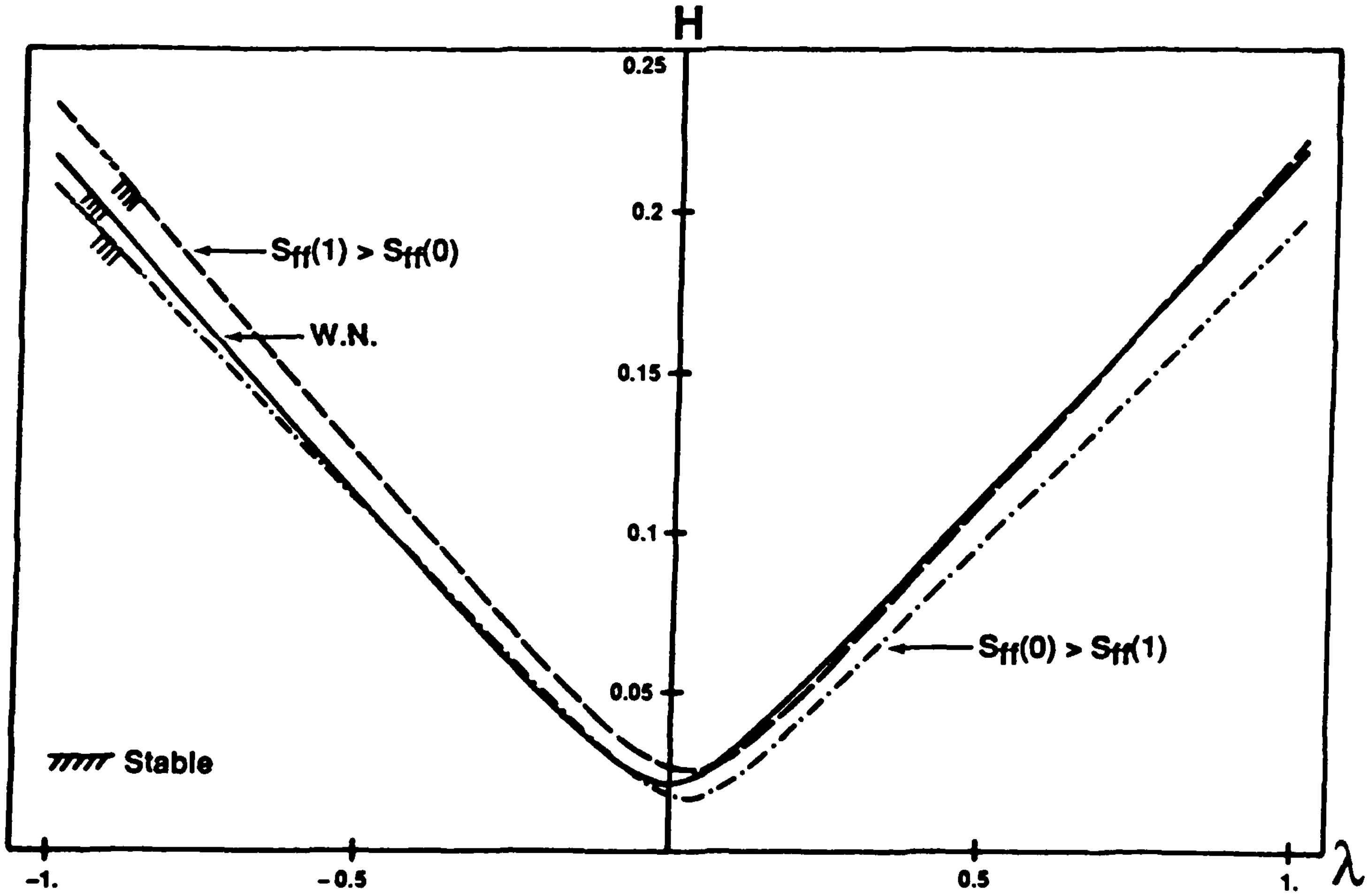


(a) First Moment Stability Boundaries

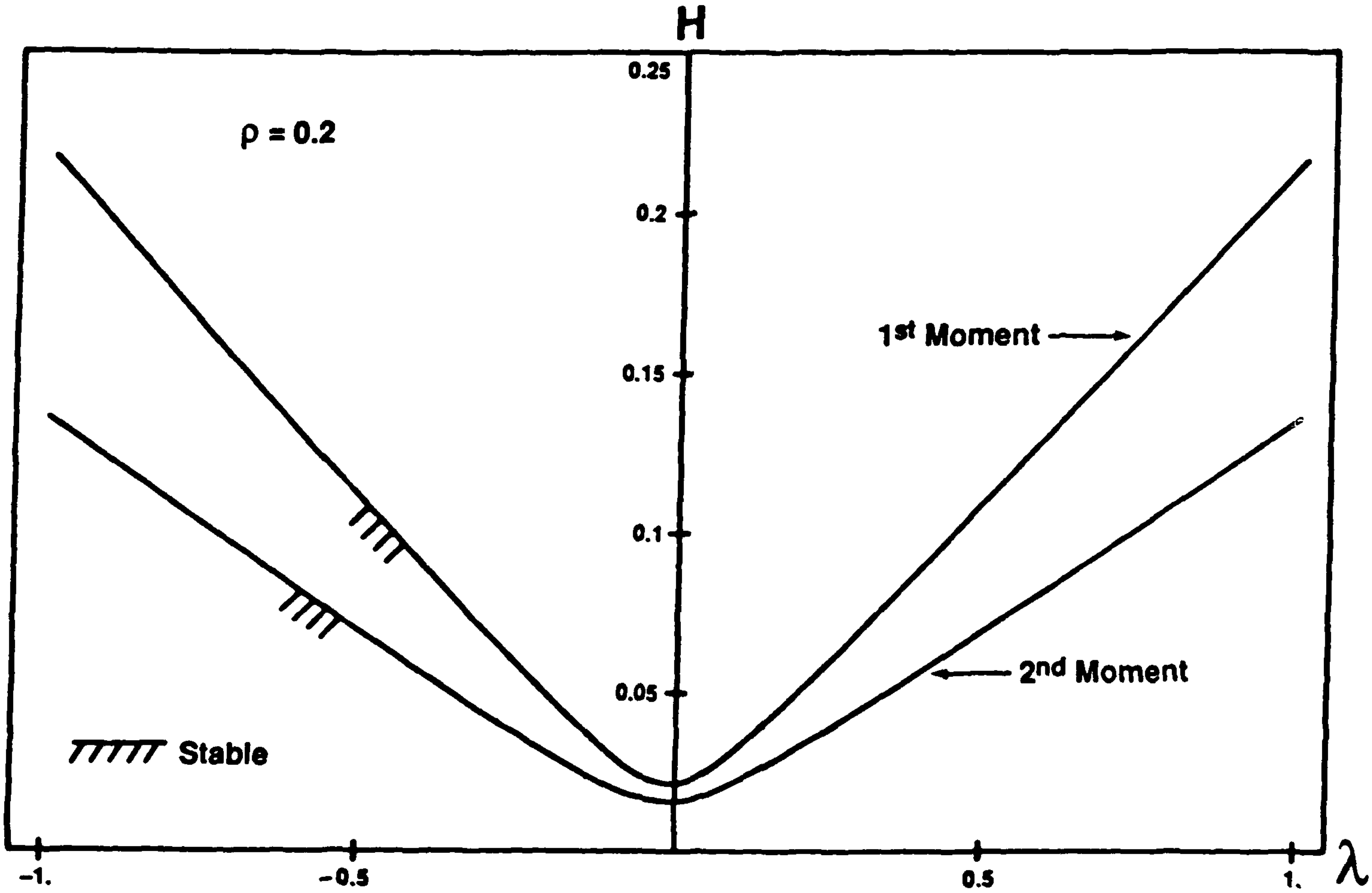


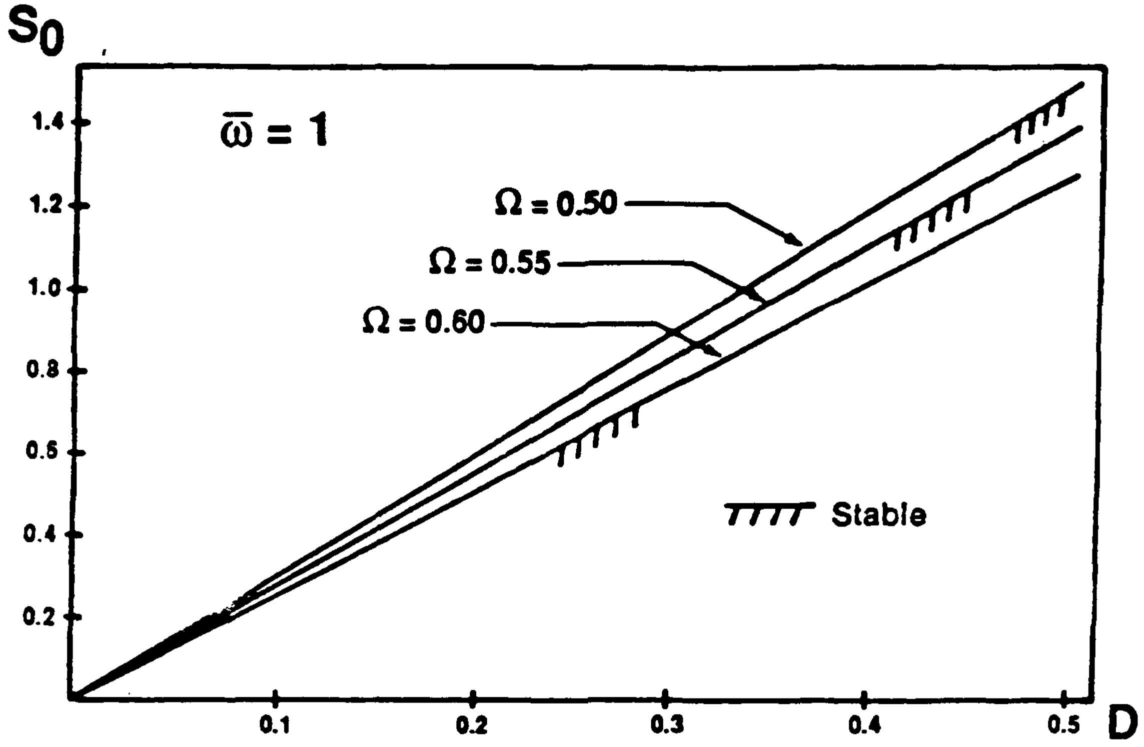
(b) Second Moment Stability Boundaries

$D = 0.1$, $S = 0.01$, $\overline{\omega} = 1$, $\Omega = 0.5$



$D = 0.1$, $S = 0.01$, $\bar{\omega} = 1$, $\Omega = 0.5$




S_0
$\bar{\omega} = 1$
$\Omega = 0.50$
$\Omega = 0.55$
$\Omega = 0.60$
Stable
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.1
0.2
0.3
0.4
0.5
D

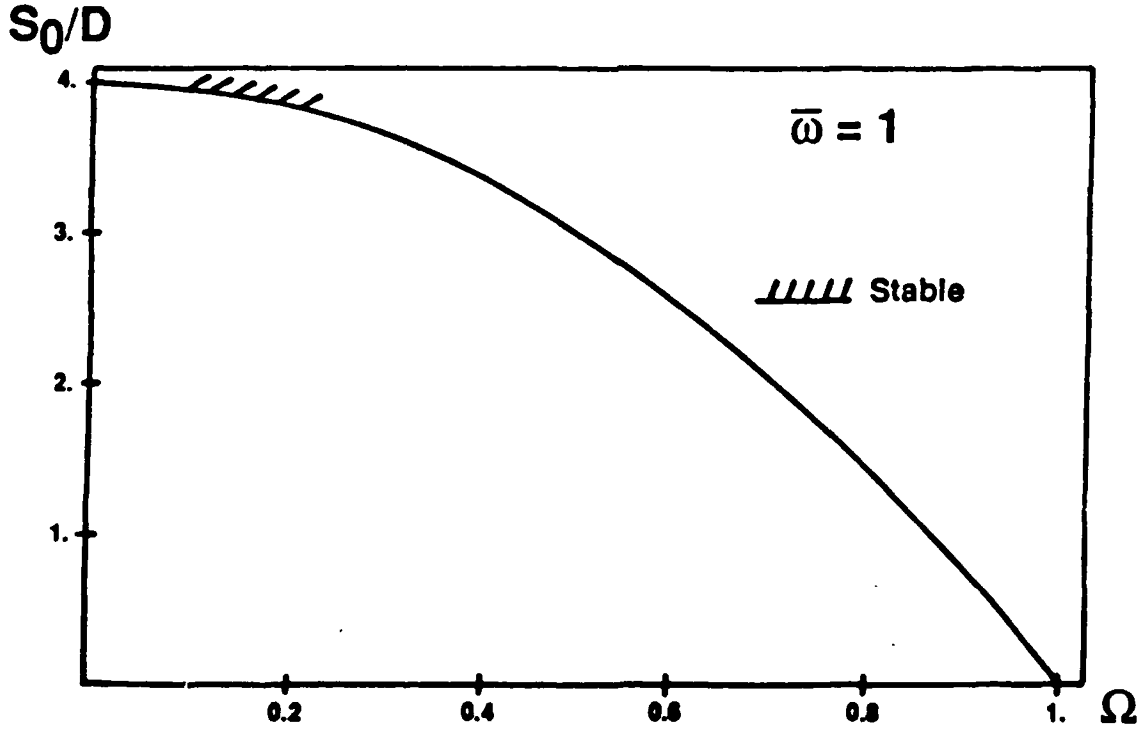

S_0/D
$\bar{\omega} = 1$
Stable
4.
3.
2.
1.
0.2
0.4
0.6
0.8
1.
Ω

Figure Captions

Figure 1(a) Rotating shaft with pulsating axial load.

(b) Stability boundaries for the deterministic undamped case.

Figure 2 First and second moment stability boundaries for $h \neq 0$ and $f(t)$ is white noise.

Figure 3 Comparison of first moment stability boundaries for white noise and non-white noise excitations.

Figure 4 Comparison of first and second moments stability boundaries for the white noise case.

Figure 5 Mean square stability boundaries for purely stochastic (white noise) excitation ($h = 0$).

Appendix - E1

I. $\kappa_1 + \kappa_2 = 1$

$$a_0 = \{[(\tfrac{1}{2} S_{11} - \tfrac{1}{2} B_{11})B_{22} + (S_{11} - B_{11})S_{12} + H^{+2}$$

$$+ \tfrac{1}{2} S_{11}B_{11} - \tfrac{1}{2} B_{11}{}^{2}]B_{22} - [(S_{11} - B_{11})S_{12} + H^{+2}$$

$$+ \tfrac{1}{2} S_{12}{}^{2} + \tfrac{1}{2} S_{11}B_{11} - \tfrac{1}{2} B_{11}{}^{2}]B_{22} - (2S_{12} - S_{11} + B_{11})H^{+2}$$

$$- (\tfrac{1}{2} S_{11} - \tfrac{1}{2} B_{11})\, B_{22}{}^{2} - S_{12}{}^{3} - \tfrac{1}{2} S_{12}{}^{2}B_{11}\}S_{00}^{+}$$

$$+ \{[(S_{11} - B_{11})S_{12} + H^{+2} + \tfrac{1}{2} S_{11}B_{11} - \tfrac{1}{2} B_{11}{}^{2}] \cdot B_{22}$$

$$+ (2S_{12} + B_{11})H^{+2} + (\tfrac{1}{4} S_{11} - \tfrac{1}{4} B_{11})\, B_{22}{}^{2}$$

$$+ (S_{11} - B_{11})S_{12}{}^{2} + (S_{11} - B_{11})S_{12}B_{11} + \tfrac{1}{4} S_{11}B_{11}{}^{2}$$

$$- \tfrac{1}{4} B_{11}{}^{3}\}\, S_{22} - [(4S_{12} - S_{11} + 2B_{11})H^{+2} + (S_{11} - \tfrac{1}{2} B_{11})S_{12}{}^{2}$$

$$+ (S_{11} - B_{11})S_{12}B_{11} + S_{12}{}^{3} + \tfrac{1}{4} S_{11}B_{11}{}^{2} - \tfrac{1}{4} B_{11}{}^{3}]\, B_{22}$$

$$+ [(\tfrac{1}{4} S_{11} - \tfrac{1}{4} B_{11})S_{22} - (\tfrac{1}{4} S_{11} - \tfrac{1}{4} B_{11})B_{22} - \tfrac{1}{4} S_{12}{}^{2}]\, \lambda^{2}$$

$$+ [(\tfrac{1}{4} S_{11} - \tfrac{1}{4} B_{11})S_{22} - (\tfrac{1}{4} S_{11} - \tfrac{1}{4} B_{11})B_{22} - \tfrac{1}{4} S_{12}{}^{2}]\, S_{00}^{+2}$$

$$- [(S_{11} - B_{11}) \cdot S_{12} + H^{+2} + \tfrac{1}{4} S_{12}{}^{2} + \tfrac{1}{2} S_{11}B_{11} - \tfrac{1}{2} B_{11}{}^{2}]\, B_{22}{}^{2}$$

$$+ [2(S_{11} - 2B_{11})\, S_{12} - 4S_{12}^{\;2} + S_{11}B_{11} - B_{11}^{\;2}]H^{+2}$$

$$- (\tfrac{1}{4} S_{11} - \tfrac{1}{4} B_{11})B_{22}^{\;3} - S_{12}^{\;4} - S_{12}^{\;3}B_{11} - \tfrac{1}{4} S_{12}^{\;2}B_{11}^{\;2}$$

$$a_1 = - \{[(\tfrac{1}{2} B_{22} + S_{12} - S_{11} + \tfrac{3}{2} B_{11})\, S_{22} - (S_{12} - \tfrac{3}{2} S_{11} + 2B_{11})B_{22}$$

$$+ (S_{11} - B_{11})S_{12} - \tfrac{1}{2} B_{22}^{\;2} + 2H^{+2} + S_{12}^{\;2} + \tfrac{1}{2} S_{11}B_{11} - \tfrac{1}{2} B_{11}^{\;2}]S^{+}_{00}$$

$$+ [(S_{12} - S_{11} + \tfrac{3}{2} B_{11})B_{22} - (2S_{11} - 3B_{11})S_{12} + \tfrac{1}{4} B_{22}^{\;2} - 2H^{+2}$$

$$+ S_{12}^{\;2} - S_{11}B_{11} + \tfrac{5}{4} B_{11}^{\;2}]\, S_{22} + \tfrac{1}{4} (S_{22} - B_{22} + S_{11} - B_{11})\lambda^2$$

$$+ \tfrac{1}{4} (S_{22} - B_{22} + S_{11} - B_{11})S^{+2}_{00} + (4H^{+2} + 3S_{12}S_{11} - 4S_{12}B_{11}$$

$$+ \tfrac{3}{2} S_{11}B_{11} - \tfrac{7}{4} B_{11}^{\;2})B_{22} + 2(4S_{12} - S_{11} + 2B_{11})H^{+2}$$

$$- (S_{12} - \tfrac{5}{4} S_{11} + \tfrac{7}{4} B_{11})B_{22}^{\;2} + (S_{11} - B_{11})S_{12}B_{11} - \tfrac{1}{4} B_{22}^{\;3}$$

$$+ 2S_{12}^{\;3} + S_{12}^{\;2}S_{11} + \tfrac{1}{4} S_{11}B_{11}^{\;2} - \tfrac{1}{4} B_{11}^{\;3}\}$$

$$a_2 = \tfrac{1}{4} \lambda^2 + \tfrac{1}{4} S^{+2}_{00} - S^{+}_{00}S_{22} + \tfrac{3}{2} S^{+}_{00}B_{22} + S^{+}_{00}S_{12} - S^{+}_{00}S_{11}$$

$$+ \tfrac{3}{2} S^{+}_{00}B_{11} - S_{22}B_{22} - 2S_{22}S_{12} + S_{22}S_{11} - 2S_{22}B_{11}$$

$$+ \tfrac{5}{4} B_{22}^{\;2} + 3B_{22}S_{12} - 2B_{22}S_{11} + \tfrac{7}{2} B_{22}B_{11} - 4H^{+2}$$

$$- 2S_{12}S_{11} + 3S_{12}B_{11} - S_{11}B_{11} + \frac{5}{4} B_{11}^2$$

$$a_3 = S_{00}^{+} - S_{22} + 2B_{22} + 2S_{12} - S_{11} + 2B_{11}$$

II. $\kappa_1 - \kappa_2 = 1$

$$a_0 = - \{[(\frac{1}{2} S_{11} - \frac{1}{2} B_{11})B_{22} - (S_{11} - B_{11})S_{12} - \bar{H}^2 + \frac{1}{2} S_{11}B_{11}$$

$$- \frac{1}{2} B_{11}{}^2]S_{22} + [(S_{11} - B_{11})S_{12} + \bar{H}^2 - \frac{1}{2} S_{12}{}^2 - \frac{1}{2} S_{11}B_{11} + \frac{1}{2} B_{11}{}^2]B_{22}$$

$$- (2S_{12} + S_{11} - B_{11})\bar{H}^2 - (\frac{1}{2} S_{11} - \frac{1}{2} B_{11})B_{22}{}^2 + S_{12}{}^3 - \frac{1}{2} S_{12}{}^2B_{11}\}S_{00}^{-}$$

$$- \{[(S_{11} - B_{11})S_{12} + \bar{H}^2 - \frac{1}{2} S_{11}B_{11} + \frac{1}{2} B_{11}{}^2]B_{22} - (2S_{12} - B_{11})\bar{H}^2$$

$$- (\frac{1}{4} S_{11} - \frac{1}{4} B_{11})B_{22}{}^2 - (S_{11} - B_{11})S_{12}{}^2 + (S_{11} - B_{11})S_{12}B_{11}$$

$$- \frac{1}{4} S_{11}B_{11}{}^2 + \frac{1}{4} B_{11}{}^3\}S_{22} - [(4S_{12} + S_{11} - 2B_{11})\bar{H}^2$$

$$+ (S_{11} - \frac{1}{2} B_{11})S_{12}{}^2 - (S_{11} - B_{11})S_{12}B_{11} - S_{12}{}^3 + \frac{1}{4} S_{11}B_{11}{}^2$$

$$- \frac{1}{4} B_{11}{}^3]B_{22} + [(\frac{1}{4} S_{11} - \frac{1}{4} B_{11})S_{22} - (\frac{1}{4} S_{11} - \frac{1}{4} B_{11})B_{22}$$

$$- \frac{1}{4} S_{12}{}^2]\lambda^2 + [(\frac{1}{4} S_{11} - \frac{1}{4} B_{11})S_{22} - (\frac{1}{4} S_{11} - \frac{1}{4} B_{11})B_{22}$$

$$- \frac{1}{4} S_{12}{}^2]\bar{S}_{00}^2 + [(S_{11} - B_{11})S_{12} + \bar{H}^2 - \frac{1}{4} S_{12}{}^2 - \frac{1}{2} S_{11}B_{11}$$

$$+ \frac{1}{2} B_{11}{}^{2}]B_{22}{}^{2} + [2(S_{11} - 2B_{11})S_{12} + 4S_{12}{}^{2} - S_{11}B_{11} + B_{11}{}^{2}]\bar{H}^{2}$$

$$- (\frac{1}{2} S_{11} - \frac{1}{4} B_{11})B_{22}{}^{3} - S_{12}{}^{4} + S_{12}{}^{3}B_{11} - \frac{1}{4} S_{12}{}^{2}B_{11}{}^{2}$$

$$a_1 = [(\frac{1}{2} B_{22} - S_{12} - S_{11} + \frac{3}{2} B_{11})S_{22} + (S_{12} + \frac{3}{2} S_{11} - 2B_{11})B_{22}$$

$$- (S_{11} - B_{11})S_{12} - \frac{1}{2} B_{22}{}^{2} - 2\bar{H}^{2} + S_{12}{}^{2} + \frac{1}{2} S_{11}B_{11} - \frac{1}{2} B_{11}{}^{2}]\bar{S}_{00}$$

$$+ [(S_{12} + S_{11} - \frac{3}{2} B_{11})B_{22} - (2S_{11} - 3B_{11})S_{12} - \frac{1}{4} B_{22}{}^{2} - 2\bar{H}^{2} - S_{12}{}^{2}$$

$$+ S_{11}B_{11} - \frac{5}{4} B_{11}{}^{2}]S_{22} - \frac{1}{4} (S_{22} - B_{22} + S_{11} - B_{11})\lambda^{2}$$

$$- \frac{1}{4} (S_{22} - B_{22} + S_{11} - B_{11})\bar{S}_{00}^{2} + (4\bar{H}^{2} + 3S_{12}S_{11} - 4S_{12}B_{11}$$

$$- \frac{3}{2} S_{11}B_{11} + \frac{7}{4} B_{11}{}^{2})B_{22} - 2(4S_{12} + S_{11} - 2B_{11})\bar{H}^{2}$$

$$- (S_{12} + \frac{5}{4} S_{11} - \frac{7}{4} B_{11})B_{22}{}^{2} + (S_{11} - B_{11})S_{12}B_{11}$$

$$+ \frac{1}{4} B_{22}{}^{3} + 2S_{12}{}^{3} - S_{12}{}^{2}S_{11} - \frac{1}{4} S_{11}B_{11}{}^{2} + \frac{1}{4} B_{11}{}^{3}$$

$$a_2 = \frac{1}{4} \lambda^{2} + \frac{1}{4} \bar{S}_{00}^{2} + \bar{S}_{00}S_{22} - \frac{3}{2} \bar{S}_{00}B_{22} + \bar{S}_{00}S_{12} + \bar{S}_{00}S_{11}$$

$$- \frac{3}{2} \bar{S}_{00}B_{11} - S_{22}B_{22} + 2S_{22}S_{12} + S_{22}S_{11} - 2S_{22}B_{11}$$

$$+ \frac{5}{4} B_{22}{}^{2} - 3B_{22}S_{12} - 2B_{22}S_{11} + \frac{7}{2} B_{22}B_{11} + 4\bar{H}^{2}$$

$$+ 2S_{12}S_{11} - 3S_{12}B_{11} - S_{11}B_{11} + \frac{5}{4} B_{11}^{\;2}$$

$$a_3 = -\left(\bar{S}_{00} + S_{22} - 2B_{22} + 2S_{12} + S_{11} - 2B_{11}\right)$$

Appendix - E2

Mean square stability condition for purely stochastic but not white excitation

$$(\zeta/\Omega^2)(\omega_1^2 - \omega_2^2)\,[(\omega_1^2 - \omega_2^2) + 4\Omega^2] >$$

$$\{(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2 / (2\Omega^2\omega_1^2)\}\, S_{ff}(2\omega_1)$$

$$+ (1/\omega_1\omega_2)[\{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) + 2\omega_1\omega_2\}\, S_{ff}(\omega_1 + \omega_2)$$

$$- \{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) - 2\omega_1\omega_2\}\, S_{ff}(\omega_1 - \omega_2)]\ ,$$

$$(\zeta/\Omega^2)(\omega_2^2 - \omega_1^2)\,[(\omega_2^2 - \omega_1^2) + 4\Omega^2] >$$

$$\{(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2 / (2\Omega^2\omega_2^2)\}\, S_{ff}(2\omega_2)$$

$$+ (1/\omega_1\omega_2)\,[\{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) + 2\omega_1\omega_2\}\, S_{ff}(\omega_1 + \omega_2)$$

$$- \{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) - 2\omega_1\omega_2\}\, S_{ff}(\omega_1 - \omega_2)\ ,$$

$$(\zeta/\Omega^2)^2(\omega_1^2 - \omega_2^2)^2[(\omega_1^2 - \omega_2^2)^2 - 16\Omega^4] + (1/\omega_1^2\omega_2^2) * [$$

$$\frac{(\bar{\omega}_1^2 - \bar{\omega}_2^2)^4}{4\Omega^4} S_{ff}(2\omega_1)S_{ff}(2\omega_2) - 4[(\bar{\omega}_1^2 + \bar{\omega}_1^2 - 2\Omega^2)^2$$

$$- 4(\omega_1\omega_2)^2]\, S_{ff}(\omega_1 + \omega_2)\, S_{ff}(\omega_1 - \omega_2)$$

$$+ \frac{(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2}{2\Omega^2}\,[(\omega_2/\omega_1)S_{ff}(2\omega_1) + (\omega_1/\omega_2)S_{ff}(2\omega_2)]$$

$$\cdot\ [\{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) + 2\omega_1\omega_2\}S_{ff}(\omega_1 + \omega_2) - \{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2)$$

$$- 2\omega_1\omega_2\} \, S_{ff}(\omega_1 - \omega_2)]\}$$

$$> (\zeta/\Omega^2)(\omega_1^2 - \omega_2^2)/(\omega_1\omega_2) \cdot \{\frac{(\bar{\omega}_1^2 - \bar{\omega}_2^2)^2}{2\Omega^2} [(\frac{\omega_1}{\omega_2})\{(\omega_1^2 - \omega_2^2) + 4\Omega^2\} S_{ff}(2\omega_2)$$

$$+ (\frac{\omega_2}{\omega_1}) \{(\omega_1^2 - \omega_2^2) - 4\Omega^2\} \, S_{ff}(2\omega_1)]$$

$$+ 2(\omega_1^2 - \omega_2^2) \, [\{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) + 2\omega_1\omega_2\} \, S_{ff}(\omega_1 + \omega_2)$$

$$- \{(\bar{\omega}_1^2 + \bar{\omega}_2^2 - 2\Omega^2) - 2\omega_1\omega_2\} S_{ff}(\omega_1 - \omega_2)]\} \, .$$

APPENDIX F

INTRODUCTION

In the 60's, rather unusual and severe bending vibrations were observed on the trans-Arabian pipe line, which at that time was a large above-ground oil pipe line supported at 20m intervals. Several analysis were carried out by Ashley et al [1], Housner [2] and Long [3], but none was able to predict the observed frequency of the vibration, nor could explain the origin of the motion except the fact that both free and forced motions due to cross-winds may create these observed phenomena. Housner [2] did, however, indicate the possibility of a dynamic instability of the pipe at certain critical velocities of the fluid which he related to the buckling of a column. These results were verified experimentally sometime later by Dodds et al [4]. This phenomenon was due to the terms that represent the inertia forces produced by the curvature of the pipe, and was not included in the equation of motion presented by Ashley et al [1]. Long [3] used the equation of motion presented by Housner and calculated the frequencies of vibration for various end conditions by a power series method.

1.1 Literature Review

It seems that the first investigators of pipes conveying fluids mentioned in the previous section are Ashley and Haviland [1] and Housner [2]. A subsequent elegant study was made by Niordson [5], which led to the same equations of motion as that obtained by Housner [2] and to essentially the same conclusions regarding stability of pipes with simply supported ends. Furthermore, Niordson presented a treatment of the problem based on shell theory and derived the beam equation as one approximation. Later,

Handelman [6] presented an analytical method in which the character of the eigenvalues of the problem is determined from the structure of the differential equation of motion without determining specific solutions.

The existence of oscillatory instability (flutter) was fully explained in two outstanding papers by Benjamin [7]. These papers deal with the dynamics of articulated pipes (consisting of rigid tubes connected by flexible joints) conveying fluids, which is a discrete representation of the continuously flexible system. In this work, he found that a cantilevered system of articulated pipes was subjected to oscillatory instability. Benjamin was the first to perceive that the dynamical problem is independent of fluid friction, and also pointed out that the buckling instability is possible in the case of a vertical cantilevered system, where gravity is operative, if the fluid is sufficiently heavy.

Gregory and Paidoussis [8] have shown theoretically and experimentally the stability of cantilevered pipes at sufficiently high velocities. The stability of tubular cantilevers conveying fluids (neglecting gravity forces) was further discussed by Nemat-Nasser et al [9] with emphasis on the effect on the stability of velocity dependent forces, such as dissipative and Coriolis forces, they showed that such forces may destabilize the system. Subsequent papers by Herrman [10] and Herrman and Nemat-Nasser [11] stressed the connection between the problem of instability of cantilever conveying fluid and the more general problem of instability of a cantilever subjected to a "follower" type force at the free end, i.e., a force retaining the same angular disposition relative to the free end in the course of small motions of the cantilever. The effect of internal pressure on the stability of pipes conveying fluid was studied both theoretically and experimentally by Naguleswaran and Williams [12] and

it was reported that pipes with both ends supported may buckle even at low velocities by the action of internal pressure. Chen [13] studied the stability of a pipe conveying fluid with the upstream end clamped and the downstream end constrained by a linear spring, so that the boundary conditions are intermediate between those of clamped-free and clamped-pinned conditions, accordingly, both buckling and oscillatory instabilities are possible in general, depending on the spring constant.

In all the studies discussed above, the flow velocity was taken to be steady. Chen [14] examined the stability of simply-supported pipes conveying fluid with a flow velocity, U, which has a time dependent harmonic component superimposed on the steady velocity U_o, such that $U = U_o(1 + \mu\cos\omega t)$. He found that parametric instabilities could happen in such cases, and also determined the boundaries of stability-instability regions, moreover, he found that parametric combination resonance are also possible. Chen obtained the equation of motion by substituting U(t) in the original equation of motion obtained for steady flow. Hence, Chen's equation of motion did not take into account the longitudinal acceleration term and, therefore, is erroneous. Paidoussis and Issid [15] considered the case of a harmonically varying flow velocity $U = U_o(1 + \mu\cos\omega t)$, rederived the pertinent equation of motion, correcting the error in Chen's formulation, and extended the analysis to boundary conditions other than simply-supported. In this study, they obtained the regions of instabilities in the (μ,ω) parameter space using the method proposed by Bolotin [16]. It should be pointed out that these authors obtained only the regions corresponding to subharmonic resonance, and wrongly concluded that combination resonance cannot be obtained using the same method. Recently, Ariaratnam and Sri Namachchivaya [17] presented an analytical

method for the stability analysis of pipe with flow velocity $U = U_o (1 + \mu\cos\omega t)$ for both subharmonic and combination resonance cases.

Non-linear analysis of flow induced planar motions was presented by Thurman and Mote [18] for a pipe with simple supported ends conveying fluid. The analysis was carried out using perturbation technique and the authors found that, in determining the natural frequencies of the system, the important of non-linear terms increase with flow velocity, so that the range of applicability of linear theory becomes more restricted as the flow velocity increases. It was also noted that as the fluid velocity increases, the effect of the longitudinal tension variation during oscillation becomes increasingly important. More work on this line was done by Holmes [19] and Rousselet and Herrman [20].

An analysis taking into account the circumferential modes especially for short pipes was made by Paidoussis and Denise [21,22]. They analyzed both cantilevered pipes and pipes with clamped ends and found that in addition to instabilities in the beam modes, instabilities in the circumferential modes are also possible, and verified these findings by experiments. Similar theoretical results were obtained later by a different analytical method by Weaver and Unny [23] for simply-supported shells. Chen and Rosenberg [24] studied the fluid-shell interaction characteristics in the small flow velocity range less than the subcritical flow velocity considering the fluid to be ideally compressible.

1.2 Scope of Present Research

Although the Hamiltonian approach given in [17,25,26] is elegant, the meaning of the physical variables are sometimes lost and the effect of damping may not be fully included, moreover, the equations of [illegible]

supported pipes conveying pulsating fluid contain nonconservative parametric excitation terms. For these reasons, a non-Hamiltonian approach is used to develop an analytical method for studying the stability and bifurcation behavior of supported pipes conveying pulsating flow.

The nonlinear dynamical system under investigation is formulated in detail in Chapter 2. In addition, various transformations are made to derive a set of equations in "standard form". The stability boundaries and bifurcation behavior of pipes in the presence of parametric excitations for the cases of subharmonic and combination resonance are discussed in Chapter 3 and 4, respectively. The numerical scheme developed for calculating the stability boundaries and bifurcation paths for large parameter values for the autonomous, averaged equation is presented in Chapter 5. Finally, Chapter 6 summarizes the conclusions of this study.

STATEMENT OF THE PROBLEM

2.1 Problem Definition and Formulation

This research investigates the transverse motion of a uniform pipe of length L, mass per unit length m and flexural rigidity EI, filled with fluid of mass per unit length M, with various support conditions . In general, the fluid flow field will be affected by the lateral vibration of the pipe, similarly the fluid interacts with the pipe itself and alters the vibrational behavior of the system. In this study, the fluid is considered to be incompressible and inviscid, flowing in a pipe of constant cross-sectional area and perimeter. Furthermore, the effects of pipe motion on the fluid are not accounted for, while the effects of fluid on the motion of pipe is considered.

The equation of motion is derived by using the energy principle. The methodology presented herein is similar to that given in [15]. For sufficient accuracy a linear moment-curvature relationship is assumed. The potential energy of deformation or, equivalently, the strain energy of the system, considering first order nonlinearities in the axial strain is given by

$$U = \frac{EA}{2} \int_0^{\ell} \left[\frac{T_o}{EA} + w' + \frac{1}{2} y'^2\right]^2 dx + \frac{EI}{2} \int_0^{\ell} y''^2 \, dx \; . \qquad (2.1)$$

where w and y are the longitudinal and transverse displacements, T_o is the externally applied tension, and prime represents differentiation with respect to x. The kinetic energy of the pipe is

$$T_1 = \frac{m}{2} \int_0^{\ell} \dot{y}^2 \, dx \quad , \qquad (2.2)$$

where dot represents differentiation with respect to time t. Furthermore, the fluid kinetic energy is

$$T_2 = \frac{M}{2} \int_0^{\ell} \{(\dot{y} + uy')^2 + [u(1 - \frac{1}{2} y^2) - \dot{C}_o]^2\} \, dx \, , \qquad (2.3)$$

where the axial contraction in the x-direction is expressed as

$$C_o = \frac{1}{2} \int_0^{x} (y'(\xi,t)^2) \, d\xi \, .$$

Benjamin [5] has shown the statement of Hamilton's principle for a pipe conveying fluid, in the absence of dissipative forces can be written as

$$\delta \int_{t_1}^{t_2} (L + M\,u^2 C_\ell) \, dt - \int_{t_1}^{t_2} Mu(\dot{y}_\ell + uy'_\ell) \, \delta y_\ell \, dt \, , \qquad (2.4)$$

where the Lagrangian $L = T_1 + T_2 - U$ and subscript ℓ represents the values of the corresponding quantities at $x = \ell$. Specifically, for supported pipes, since $C_\ell = 0$ and $y_\ell = 0$ equation (2.4) can be reduced to

$$\delta \int_{t_1}^{t_2} L dt = 0 \, . \qquad (2.5)$$

Substituting equations (2.1), (2.2) and (2.3) into equation (2.5), one obtains

$$\delta \int_{t_1}^{t_2} \frac{1}{2} Mu^2 \ell dt + \delta \int_{t_1}^{t_2} \int_0^{\ell} [\frac{(M + m)}{2} \dot{y}^2 + Mu(\dot{y}y' + \frac{1}{2} uy'^2)$$

$$- Mu\,\dot{C} - \frac{EA}{2} (\frac{T_o}{EA} + W' + \frac{1}{2} y')^2 - \frac{EI}{2} y''^2] dxdt = 0 \qquad (2.6)$$

By applying the usual variational techinques to Eq. (2.6) the equation of small lateral motions is obtained as

$$(M + m)\ddot{y} + 2Mu\,\dot{y}' + Mu^2 y'' - T_o y'' + M\dot{u}(\ell - x)y'' + EIy^{IV}$$

$$- EA\left[w' + \frac{1}{2}y'^2)y'\right]' = 0\,. \tag{2.7}$$

The corresponding boundary conditions for pinned-pinned and clamped-clamped pipes can be written respectively as

$$y(o,t) = y(\ell,t) = 0\ ,\quad y''(o,t) = y''(\ell,t) = 0\ ;$$

$$y(o,t) = y(\ell,t) = 0\ ,\quad y'(o,t) = y'(\ell,t) = 0\ ,$$

one can define an average axial strain $\varepsilon_o(t)$ as

$$\varepsilon_o(t) = \frac{1}{\ell}\int_o^\ell (w' + \frac{1}{2}y'^2)dx = \frac{1}{2\ell}\int_o^\ell (y')^2\,dx\,.$$

Substituting the above equation in Eq. (2.7) yields the equation of the transverse motion as

$$(M + m)\ddot{y} + 2Mu\dot{y}' + \left[M\dot{u}(\ell - x) + Mu^2\right.$$

$$\left. - T_o - \frac{EA}{2\ell}\int_o^\ell (y')^2\,dx\right]y'' + EIy^{IV} = 0\,. \tag{2.8}$$

By defining the following non-dimensional quantities

$$\kappa = \frac{A\ell}{2I}\,,\quad \xi = x/\ell\,,\quad \eta = y/\ell\,,\quad \bar{t} = \left(\frac{EI}{M + m}\right)^{1/2}\frac{t}{\ell^2}\,,$$

$$\bar{T} = \frac{T_o \ell^2}{EI}\ , \quad \bar{u} = \left(\frac{M}{EI}\right)^{1/2} u\ell\ , \ Mr = \left(\frac{M}{M + m}\right)^{1/2}\ , \quad \bar{E}^* = \frac{E^*}{\ell^2}\left(\frac{I}{E(M + m)}\right)^{1/2}\ ,$$

and incorporating damping terms, the dimensionless equation of motion is obtained as

$$\bar{E}^* \dot{\eta}^{IV} + \eta^{IV} + [\bar{u}^2 - \bar{T} + (1-\xi)\ Mr\ \dot{\bar{u}}]\ \eta'' + 2Mr\bar{u}\dot{\eta}' + \ddot{\eta}$$

$$- \kappa\eta'' \int_o^\ell (\eta')^2\ d\xi = 0\ , \tag{2.9}$$

where the dot and prime of the above equation represent the differentiation with respect to new time $\bar{t}$ and ξ, respectively and E* is the coefficient of internal dissipation which is assumed to be viscoelastic and of the Kelvin-Voigt type. Furthermore, the nonlinear damping terms such as

$$E^* \frac{A}{\ell} \left[\int_o^\ell (\dot{y}\ \dot{y}')\ dx\right] y^2\ ,$$

are assumed to be small. The fluid velocity is assumed to be harmonically varying and given by $\bar{u} = \bar{u}_o(1 + \mu\cos\nu\bar{t})$, where $\bar{u}_o$ is the mean velocity, ν is the frequency of the parametric excitation and μ is the amplitude of the periodic perturbation which is assumed to be small and of the order ε. Thus, one can approximate $\bar{u}^2 = \bar{u}_o^2\ (1 + 2\mu\cos\nu\bar{t})$. The discrete equations of motion corresponding to equation (2.9) are obtained by the application of the Ritz-Galerkin method. Thus, approximating the transverse motion by

$$\eta(\xi,\bar{t}) = \sum_{r=1}^{n} \hat{\Phi}_r\ (\xi) q_r(\bar{t})\ ,$$

where $q_r(\bar{t})$ are the generalized coordinates and $\hat{\Phi}_r(\xi)$ are the eigenfunction corresponding to the free undamped vibration of a beam satisfying all the boundary conditions, the discrete equations are evaluated as

$$I\ddot{\underline{q}} + 2M_r\bar{u}_o B\dot{\underline{q}} + [\Lambda + (\bar{u}_o^2 - \bar{T})C]\underline{q} + \frac{\partial U}{\partial \underline{q}}$$

$$= \varepsilon\{h\nu D_1\underline{q}\ \sin\nu\bar{t} - h[D_2\underline{q} + D_3\dot{\underline{q}}]\cos\nu\bar{t} - E^*\Lambda\dot{\underline{q}}\}\ , \qquad (2.10)$$

where $\partial U/\partial q$ represents the nonlinear terms, $\mu = \varepsilon h$, $\bar{E}^* = \varepsilon E^*$, $\Lambda = \mathrm{diag}\{\lambda_1^4, \lambda_2^4, \ldots .\ \lambda_n^4\}$ the λ_i's being the in vacuo eigenvalues (with no fluid) of the system. D_1 D_2 and D_3 are constant nxn matrices defined as

$$D_1 = M_r\bar{u}_o(C-D)\ ,\quad D_2 = 2\bar{u}_o C\ ,\quad D_3 = 2M_r\bar{u}_o B\ ,$$

where B, C and D are constant nxn matrices, whose elements b_{rs}, c_{rs} and d_{rs}, respectively, involve integrals of the eigenfunctions and are given by

$$b_{rs} = \int_0^1 \hat{\Phi}_r(\xi)\ \hat{\Phi}_s(\xi)\ d\xi\ ,$$

$$c_{rs} = \int_0^1 \hat{\Phi}_r''(\xi)\ \hat{\Phi}_s(\xi)\ d\xi\ ,$$

$$d_{rs} = \int_0^1 \xi\hat{\Phi}_r(\xi)\ \hat{\Phi}_s''(\xi)\ d\xi\ .$$

Furthermore, this study considers only geometric nonlinearities of the type given by

$$U = \kappa\ \hat{C}_{ijk\ell}\ q_i q_j q_k q_\ell\ ,\quad \text{where}\quad \hat{C}_{ijk\ell} = C_{ij}C_{k\ell}\ ,$$

k = (AL)/(4I) for pinned-pinned support condition and k = (AL)/(2I) for the

rest of the support conditions. Thus, equation (2.10) describes the parametrically excited motion of gyroscopic, discrete, nonlinear mechanical systems with n degrees of freedom about the equilibrium configuration $\underset{\sim}{q} = 0$.

2.2 Transformation to Standard Form

For the purpose of studying the stability and bifurcation behavior of supported pipes, the system is investigated by restricting it to a two-mode discrete equations of the form

$$\ddot{q}_1 - 2G\dot{q}_2 + K_{11}q_1 = \varepsilon F_1(\underset{\sim}{q},\dot{\underset{\sim}{q}},\bar{t}) , \tag{2.11a}$$

$$\ddot{q}_2 + 2G\dot{q}_1 + K_{22}q_2 = \varepsilon F_2(\underset{\sim}{q},\dot{\underset{\sim}{q}},\bar{t}) , \tag{2.11b}$$

where

$$G = M_r\bar{u}_o B_{12} ,$$

$$K_{11} = \lambda_1^4 + (\bar{u}_o^2 - \bar{T})\, C_{11} ,$$

$$K_{22} = \lambda_2^4 + (\bar{u}_o^2 - \bar{T})\, C_{22} .$$

In the above equation, dot represents the differentiation w.r.t new time τ, where $\omega_o\tau = \nu\bar{t}$, $\nu = \omega_o(1-\varepsilon\lambda)$, and λ is the detuning parameter. The expressions for F_1, F_2 contain nonlinear terms, damping terms and detuning parameter of the system. It is obvious that the equations (2.11) do not have an exact solution. It is, therefore, important to use an approximate method which offers a very elegant summary of results. One can use, for this purpose, the method of slowly varying phase and amplitude which takes advantage of the well known process of averaging with respect to time τ. Now, in order to apply the method of averaging, one must transform Eqs.

(2.11) to a suitable "standard form". This is achieved by means of a transformation based on the solution of the unperturbed system corresponding to $\varepsilon = 0$ of Eqs. (2.11), i.e., by assuming

$$q_1 = Q_{11}e^{i\omega_1\tau} + Q_{12}e^{-i\omega_1\tau} + Q_{13}e^{i\omega_2\tau} + Q_{14}e^{-i\omega_2\tau} ,$$

$$q_2 = Q_{21}e^{i\omega_1\tau} + Q_{22}e^{-i\omega_1\tau} + Q_{23}e^{i\omega_2\tau} + Q_{14}e^{-i\omega_2\tau} ,$$

$$= \alpha_1 Q_{11}e^{i\omega_1\tau} - \alpha_1 Q_{12}e^{-i\omega_1\tau} + \alpha_2 Q_{13}e^{i\omega_2\tau} - \alpha_2 Q_{14}e^{-i\omega_2\tau} ,$$

where ω_i and Q_{ij} are the eigenvalues and the i's eigenvector of the unperturbed system, respectively, α_i is the mode ratio of the unperturbed system, one can obtain

$$q_1 = z_1\sin(\omega_1\tau + \phi_1) + z_2\sin(\omega_2\tau + \phi_2)$$

$$= z_1\sin\Phi_1 + z_2\sin\Phi_2 , \tag{2.12}$$

$$\dot{q}_1 = z_1\omega_1\cos\Phi_1 + z_2\omega_2\cos\Phi_2 . \tag{2.13}$$

Similarly

$$q_2 = \alpha_1 z_1\cos(\omega_1\tau + \phi_1) + \alpha_2 z_2\cos(\omega_2\tau + \phi_2)$$

$$= \alpha_1 z_1\cos\Phi_1 + \alpha_2 z_2\cos\Phi_2 , \tag{2.14}$$

$$\dot{q}_2 = - z_1\alpha_1\omega_1\sin\Phi_1 - z_2\alpha_2\omega_2\sin\Phi_2 , \tag{2.15}$$

where $z_1 = \sqrt{2}\,(Q_{11}^2 + Q_{12}^2)^{1/2}$, $z_2 = \sqrt{2}\,(Q_{13}^2 + Q_{14}^2)^{1/2}$,

$$\alpha_1 = \frac{\omega_1^{\,2} - K_{11}}{2G\omega_1} , \quad \alpha_2 = \frac{\omega_2^{\,2} - K_{11}}{2G\omega_2} .$$

By assuming both Z_i and ϕ_i to be dependent on τ Eqs. (2.13) and (2.15) become

$$\dot{Z}_1 \sin\Phi_1 + \dot{Z}_2 \sin\Phi_2 + \dot{\phi}_1 Z_1 \cos\Phi_1 + \dot{\phi}_2 Z_2 \cos\Phi_2 = 0 \,, \tag{2.16}$$

$$\alpha_1 \dot{Z}_1 \cos\Phi_1 + \alpha_2 \dot{Z}_2 \cos\Phi_2 - \alpha_1 \dot{\phi}_1 Z_1 \sin\Phi_1 - \alpha_2 \dot{\phi}_2 Z_2 \sin\Phi_2 = 0 \,, \tag{2.17}$$

substituting for $\ddot{q}_i, \dot{q}_i, q_i$ in Eqs. (2.11) yield

$$\omega_1 (\dot{Z}_1 \cos\Phi_1 - \dot{\phi}_1 Z_1 \sin\Phi_1) + \omega_2 (\dot{Z}_2 \cos\Phi_2 - \dot{\phi}_2 Z_2 \sin\Phi_2) = \varepsilon F_1 \,, \tag{2.18}$$

and

$$- \omega_1 \alpha_1 (\dot{Z}_1 \sin\Phi_1 + Z_1 \dot{\phi}_1 \cos\Phi_1) - \omega_2 \alpha_2 (\dot{Z}_2 \sin\Phi_2 + \dot{\phi}_2 Z_2 \cos\Phi_2) = \varepsilon F_2 , \tag{2.19}$$

respectively, the above Eqs. (2.16) - (2.19) can be written in the matrix form as

$$\begin{bmatrix} \sin\Phi_1 & \cos\Phi_1 & \sin\Phi_2 & \cos\Phi_2 \\ \alpha_1 \cos\Phi_1 & -\alpha_1 \sin\Phi_1 & \alpha_2 \sin\Phi_2 & -\alpha_2 \sin\Phi_2 \\ \omega_1 \cos\Phi_1 & -\omega_1 \sin\Phi_1 & \omega_2 \cos\Phi_2 & -\omega_2 \sin\Phi_2 \\ -\omega_1 \alpha_1 \sin\Phi_1 & -\omega_1 \alpha_1 \cos\Phi_1 & -\omega_2 \alpha_2 \sin\Phi_2 & -\omega_2 \alpha_2 \cos\Phi_2 \end{bmatrix} \begin{Bmatrix} \dot{Z}_1 \\ \dot{\phi}_1 Z_1 \\ \dot{Z}_2 \\ \dot{\phi}_2 Z_2 \end{Bmatrix} = \varepsilon \begin{Bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \end{Bmatrix} \tag{2.20}$$

Premultiplying Eq. (2.20) by the matrix S given as

$$S = \frac{1}{\Delta} \begin{bmatrix} \omega_2 \alpha_2 \Delta_2 \sin\Phi_1 & \omega_2 \Delta_1 \cos\Phi_1 & -\alpha_2 \Delta_1 \cos\Phi_1 & \Delta_2 \sin\Phi_1 \\ \omega_2 \alpha_2 \Delta_2 \cos\Phi_1 & -\omega_2 \Delta_1 \sin\Phi_1 & \alpha_2 \Delta_1 \sin\Phi_1 & \Delta_2 \cos\Phi_1 \\ -\omega_1 \alpha_1 \Delta_2 \sin\Phi_2 & -\omega_1 \Delta_1 \cos\Phi_2 & \alpha_1 \Delta_1 \cos\Phi_2 & -\Delta_2 \sin\Phi_2 \\ -\omega_1 \alpha_1 \Delta_2 \cos\Phi_2 & \omega_1 \Delta_1 \sin\Phi_2 & -\alpha_1 \Delta_1 \sin\Phi_2 & \Delta_2 \cos\Phi_2 \end{bmatrix}$$

where $\Delta_1 = \omega_2 \alpha_2 - \omega_1 \alpha_1$, $\Delta_2 = \omega_2 \alpha_1 - \omega_1 \alpha_2$, $\Delta = \Delta_1 \cdot \Delta_2$,

yields

$$\begin{Bmatrix} \dot{Z}_1 \\ Z_1\dot{\phi}_1 \\ \dot{Z}_2 \\ Z_2\dot{\phi}_2 \end{Bmatrix} = \varepsilon S \begin{Bmatrix} 0 \\ 0 \\ F_1 \\ F_2 \end{Bmatrix} \tag{2.21}$$

Equation (2.21) are now in the standard form and are exactly equivalent to the original equations of motion (2.10). By using the method of averaging, which is a first approximation of an asymptotic method, the averaged equations corresponding to equation (2.21) are written symbolically as

$$d\bar{z}/d\tau = \varepsilon \underset{t}{M}\left(\chi_z\right) \quad , \quad d\bar{\phi}/d\tau = \varepsilon \underset{t}{M}\left(\chi_\phi\right) . \tag{2.22}$$

where the averaging operator is defined as $\underset{t}{M}(\cdot) = \lim_{T \to \infty} T^{-1} \int_0^T (\cdot)d\tau$, and the integration is performed over explicit time τ. According to the mathematical basis of the method of averaging, if the averaged equations (2.22) have solution $\bar{z}_o(\tau)$ and $\bar{\phi}_o(\tau)$ then the solution of Eqs. (2.21) will remain in a small neighborhood of $\bar{z}_o(t)$ and $\bar{\phi}_o(t)$ for all time since the right-hand sides of equations (2.21) are periodic. Furthermore, the stability of the averaged system implies the stability of the solutions of equations (2.21) and the averaged equations (2.22) are accurate only in the first approximation, i.e., $z = \bar{z} + O(\varepsilon)$, $\phi = \bar{\phi} + O(\varepsilon)$. Now, by applying the averaging operator to Eqs. (2.21), one obtains a set of averaged equations in the presence of parametric resonance which occurs over specific ranges of values of ω_r in the vicinity of $2\omega_r = m\nu$ and $|\omega_r \pm \omega_s| = m\nu$, $m = 1,2,\ldots.$ where ω_r and ν are the natural and exciting

frequencies. The autonomous, averaged equations of the nonlinear system for the predominant cases (1) $2\omega_r = \omega_o$, subharmonic parametric resonance, and (2) $|\omega_r \pm \omega_s| = \omega_o$, combination parametric resonance are examined in the following chapters.

SUBHARMONIC PARAMETRIC RESCNANCE

3.1 Averaged Equations and General Results

The averaged equations of motion in the first approximation for $\omega_o = 2\omega_r$, r = 1,2 are given in Appendix A and expressed as

$$\dot{Z}_r = \varepsilon\{hZ_r[U_{rr}\sin(2\phi_r) + V_{rr}\cos(2\phi_r)] + E^*\xi_r Z_r\} ,$$
$$Z_r\dot{\phi}_r = \varepsilon Z_r\{h[U_{rr}\cos(2\phi_r) - V_{rr}\sin(2\phi_r)] + N_r - \lambda D_r\} , \tag{3.1}$$

where the terms U_{rr} and V_{rr} are defined in Appendix B-I and the remaining terms in Eqs. (3.1) are defined in Appendix B-II. Since the r.h.s. of the above equations contain $1/(4\Delta)$ as shown in Appendix B, the time is reversed by introducing a new time $T = -\tau$. Now, putting $\tan\theta = V_{rr}/U_{rr}$, $R_r = (U_{rr}^2 + V_{rr}^2)^{1/2}$ and $\hat{\phi}_r = \phi_r + \theta/2$ in the above equations yields

$$\frac{dZ_r}{dT} = -\varepsilon Z_r[hR_r\sin(2\hat{\phi}_r) + E^*\xi_r] , \tag{3.2a}$$

$$\frac{d\phi_r}{dT} = -\varepsilon[hR_r\cos(2\hat{\phi}_r) + (N_r - \lambda D_r)] . \tag{3.2b}$$

The stationary states are determined by setting $dZ_r/dT = 0$ and $d\phi_r/dT$ = constant in Eqs. (3.2), which, apart from the trivial solution $Z_r = 0$, yields an amplitude-frequency (a-f) relationship,

$$\lambda = \pm D_r^{-1}[h^2R_r^2 - (E^*\xi_r)^2]^{1/2} + D_r^{-1}N_r . \tag{3.3}$$

Equations (3.3) represent the positive and negative nontrivial solutions, which are associated with $\cos 2\hat{\phi}_{ro} = -\beta_r^{1/2}/(R_r h)$ and $\cos 2\hat{\phi}_{ro} = \beta_r^{1/2}/(R_r h)$ respectively, where $\beta_r = (h^2R_r^2 - (E^*\xi_r)^2)$. By

making use of the expression for λ, Eqs. (3.3) can be rewritten as

$$\nu = \omega_o \mp 2[h^2 R_r{}^2 - (E^*\xi_r)^2]^{1/2} - 2Z_{ro}^2 P_r \tag{3.4}$$

It may be noted in Eqs. (3.4) that the negative sign corresponds to the positive nontrivial solution and the positive sign corresponds to the negative nontrivial solution.

3.2 Stability and Bifurcation Analysis of Trivial Solution

In order to consider the stability of the trivial solution, Eqs. (3.2) are transformed from Z_r, ϕ_r to new variables X_r, $\hat{X}_r$ by means of the transformation

$$\hat{X}_r = Z_r \sin\hat{\phi}_r \ , \qquad X_r = Z_r \cos\hat{\phi}_r \ .$$

This procedure yields

$$\begin{aligned}
\frac{dX_r}{dT} &= -\epsilon Z_r[hR_r\sin(2\phi_r) + E^*\xi_r]\cos\phi_r \\
&\quad + \epsilon Z_r[hR_r\cos(2\phi_r) + N_r - \lambda D_r]\sin\phi_r \\
&= \epsilon\{E^*\xi_r X_r + (hR_r + \lambda D_r)\hat{X}_r - N_r\hat{X}_r\} \ , \\
\frac{d\hat{X}_r}{dT} &= -\epsilon Z_r[hR_r\sin(2\phi_r) + E^*\xi_r]\sin\phi_r \\
&\quad - \epsilon Z_r[hR_r\cos(2\phi_r) + N_r - \lambda D_r]\cos\phi_r \\
&= -\epsilon\{(hR_r - \lambda D_r)X_r + E^*\xi_r\hat{X}_r + N_x X_r\} \ .
\end{aligned} \tag{3.5}$$

It is evident that the linear part of Eq. (3.5) corresponds to the linear variational equations about $Z_r = 0$, whose solutions are proportional to $\exp(\bar{\rho}T)$, where $\bar{\rho} = \rho/\epsilon$ is given by

$$\bar{\rho}_{1,2} = -E^*\xi_r \pm (h^2R_r^2 - \lambda^2D_r^2)^{1/2} .$$

Therefore, the trivial solution is asymptotically stable if the following conditions hold:

$$\xi_r > 0 \ , \ |1 - \nu/\omega_o| > 2[h^2R_r^2 - (E^*\xi_r)^2]^{1/2} \ , \ (\omega_o = 2\omega_r) \ . \qquad (3.6a)$$

However, for the undamped case

$$\rho_{1,2} = \pm\ [(hR_r + \lambda D_r)(hR_r - \lambda D_r)]^{1/2}$$

and the stability cinditions get simplified to

$$\nu/\omega_o < 1 - 2\mu R_r \quad , \quad \nu/\omega_o > 1 + 2\mu R_r \qquad (3.6b)$$

It is shown that for the undamped case the trivial solution, which is stable in the Lyapunov sense, loses its stability at $\lambda D_r = \pm hR_r$ due to double zero eigenvalues. Thus, for $E^* = 0$ and $N_r = (X_r^2 + \hat{X}_r^2)P_r$, introducing a new time $\bar{T} = \epsilon T$ and linear transformations $X_r = U_r + V_r$, $\hat{X}_r = -V_r/(2R_r)$ and $X_r = -V_r/(2R_r)$, $\hat{X}_r = U_r + V_r$, for the cases $\lambda D_r = +hR_r$ and $\lambda D_r = -hR_r$, respectively, Eq. (3.5) yields

$$\frac{dU_r}{d\bar{T}} = V_r \mp [(2R_rP_r)U_r{}^3 + (6R_rP_r + \frac{P_r}{2R_r})\ U_r^2V_r$$

$$+ \left(6R_rP_r + \frac{3P_r}{2R_r}\right) U_rV_r^{\,2} + \left(2R_rP_r + \frac{P_r}{R_r} + \frac{P_r}{8R_r^{\,3}}\right) V_r^{\,3}]$$

$$= V_r \mp [a_3U_r^{\,3} + a_2U_r^{\,2}V_r + a_1U_rV_r^2 + a_0V_r^{\,3}]$$

$$\frac{dV_r}{d\bar{T}} = \pm\, [(2R_rP_r)U_r^{\,3} + (6R_rP_r)U_r^{\,2}V_r + \left(6R_rP_r + \frac{P_r}{2R_r}\right) U_rV_r^{\,2}$$

$$+ \left(2R_rP_r + \frac{P_r}{2R_r}\right) V_r^{\,3}]$$

$$= \pm\, [b_3U_r^{\,3} + b_2U_r^{\,2}V_r + b_1U_rV_r^{\,2} + b_0V_r^{\,3}]$$

The normal form of the above equations are computed with the aid of the following near-identity transformations

$$U_r = \eta_r + a_0\eta_r\xi_r^{\,2} + \left(\frac{a_1 + b_0}{2}\right) \eta_r^2\xi_r + \left(\frac{2a_2 + b_1}{6}\right) \eta_r^3$$

$$V_r = \xi_r + b_0\eta_r\xi_r^{\,2} + \left(\frac{b_1}{2}\right) \eta_r^{\,2}\xi_r - a_3\eta_r^{\,3}$$

$$\frac{d\eta_r}{d\bar{T}} = \xi_r \quad , \quad \frac{d\xi_r}{d\bar{T}} = \pm\, 2R_rP_r\eta_r^{\,3} \tag{3.7}$$

In the above equations, positive and negative signs correspond to the cases $\lambda = +\,2hR_r$ and $\lambda = -\,2hR_r$, respectively. It is evident from Eqs (3.7) that the point corresponding to $\nu/\omega_0 = 1 + 2hR_r$ ($\lambda = -\,2hR_r$) is unstable. Moreover, two parameters - μ_1 and μ_2 are needed to completely

unfold the singularity [27], and these parameters usually represent the determinant and the trace of the linear operator when $E^* \neq 0$, $\lambda D_r \neq \mp hR_r$. However, in this study E^* is identically zero and the damping is generally fixed. Thus by introducing $\mu_1 = [h^2 R_r^{\,2} - (1 - \nu/\omega_o)^2 D_r^{\,2}]$ one obtains

$$\frac{d\eta_r}{d\bar{T}} = \xi_r \quad , \quad \frac{d\xi_r}{d\bar{T}} = \mu_1 \eta_r \pm 2R_r P_r \eta_r^{\,3} \ .$$

The fixed points are given by

$$\eta_r = 0 \ , \ \xi_r = 0; \ \eta_r = \left(\frac{\mp \mu_1}{2R_r P_r}\right)^{1/2} \ , \ \xi_r = 0; \ \eta_r = -\left(\frac{\mp \mu_1}{2R_r P_r}\right)^{1/2} \ , \ \xi_r = 0 \ ,$$

and the nontrivial fixed points exist only for the following cases:

1. If $P_r < 0$, then for the cases $\nu/\omega_o = 1 - 2hR_r$ and $\nu/\omega_o = 1 + 2hR_r$ nontrivial fixed points exist for $\mu_1 > 0$ and $\mu_1 < 0$, respectively.
2. If $P_r > 0$, then for the cases $\mu/\omega_o = 1 - 2hR_r$ and $\nu/\omega_o = 1 + 2hR_r$ nontrivial fixed points exist for $\mu_1 < 0$ and $\mu_1 > 0$, respectively.

3.3 Stability of the Nontrivial Solution

The stability of the "global" nontrivial solution (3.3) is investigated by examining the linear variational equation of the averaged equations (3.2) about the nontrivial solution. Letting $Z_r = Z_{ro} + X_r$ and $\phi_r = \phi_{ro} + Y_r$, (i.e., $\hat{\phi}_{ro} = \phi_{ro} + \theta/2$) the linear variational equation is obtained as

$$\frac{dX_r}{dT} = -\varepsilon[(hR_r \sin 2\hat{\phi}_{ro} + E^*\xi_r)X_r + (2hZ_{ro}R_r \cos 2\hat{\phi}_{ro})Y_r] \ , \qquad (3.8)$$

$$\frac{dY_r}{dT} = -\varepsilon[2Z_{ro}P_r X_r - 2hR_r \sin 2\hat{\phi}_{ro} Y_r] \ .$$

Using the a-f relation corresponding to $\cos 2\hat{\phi}_{ro} = \pm \frac{\beta_r^{1/2}}{R_r h}$, the eigenvalues of the characteristic equation corresponding to Eq. (3.8) are obtained as

$$\rho_{1,2} = - E^*\xi_r \pm [(- E^*\xi_r)^2 \pm 4Z_{ro}{}^2 P_r \beta_r{}^{1/2}]^{1/2} . \qquad (3.9)$$

The positive sign within the square brackets in Eq. (3.9) corresponds to the positive nontrivial solution of Eq. (3.4). It is evident from Eq. (3.9) that for $P_r < 0$, the positive nontrivial solution is stable, while the negative nontrivial solution is unstable. The opposite results prevail for $P_r > 0$. For the undamped case i.e., $E^* = 0$, the equation of motion reduces to

$$\frac{dZ_r}{d\phi_r} = \frac{Z_r[hR_r \sin 2\hat{\phi}_r]}{[hR_r \cos 2\hat{\phi}_r - \lambda D_r + 2Z_r{}^2 P_r]} .$$

Now, by integrating the above expression, and putting $X_r = \sqrt{Z_r} \cos\hat{\phi}_r$ and $Y_r = \sqrt{Z_r} \sin\hat{\phi}_r$, one obtains

$$9hR_r(X_r - Y_r)^2 + 4(X_r{}^2 + Y_r{}^2)P_r - 6\lambda D_r(X_r{}^2 + Y_r{}^2) = \text{Const.} \qquad (3.10)$$

With the help of Eq. (3.10) one can obtain various phase portraits for different values of λ and h, and thus the stability.

COMBINATION PARAMETRIC RESONANCE

4.1 Averaged Equations and Stability Boundary

The bifurcation in the presence of combination resonance, which exists under the conditions $|\omega_1 \pm \omega_2| = \omega_0$ is studied in this chapter. For the case of $|\omega_1 + \omega_2| = \omega_0$, by applying the averaging operator to Eq. (2.21), one can obtain the averaged equations of motion in the first approximation given in Appendix A and express as

$$\begin{aligned}
\dot{Z}_1 &= \varepsilon\{hZ_2[U_{12}\sin(\phi_1 + \phi_2) + V_{12}\cos(\phi_1 + \phi_2)] + Z_1E^*\xi_1\} , \\
\dot{Z}_2 &= \varepsilon\{hZ_1[U_{21}\sin(\phi_1 + \phi_2) + V_{21}\cos(\phi_1 + \phi_2)] + Z_2E^*\xi_2\} , \qquad (4.1) \\
Z_1\dot{\phi}_1 &= \varepsilon\{hZ_2[U_{12}\cos(\phi_1 + \phi_2) - V_{12}\sin(\phi_1 + \phi_2)] + (N_1 - \lambda D_1)Z_1\} , \\
Z_2\dot{\phi}_2 &= \varepsilon\{hZ_1[U_{21}\cos(\phi_1 + \phi_2) - V_{21}\sin(\phi_1 + \phi_2)] + (N_2 - \lambda D_2)Z_2\} ,
\end{aligned}$$

where the quantities U_{12}, U_{21}, V_{12}, and V_{21} are defined in Appendix B-I. Since it can be shown from the numerical calculation that $U_{12}V_{21} - U_{21}V_{12} = 0$ and the individual terms are nonzero, one can take $U_{12}/V_{12} = U_{21}/V_{21}$. Introducing a new time $T = -\tau$ as before and putting $V_{12}/U_{12} = \tan\theta$, the equation of motion can be reduced to

$$\begin{aligned}
\frac{dZ_1}{dT} &= -\varepsilon\{hZ_2R_{12}\sin(\phi_1 + \phi_2 + \theta) + Z_1E^*\xi_1\} , \\
\frac{dZ_2}{dT} &= -\varepsilon\{hZ_1R_{21}\sin(\phi_1 + \phi_2 + \theta) + Z_2E^*\xi_2\} , \\
Z_1\frac{d\phi_1}{dT} &= -\varepsilon\{hZ_2R_{12}\cos(\phi_1 + \phi_2 + \theta) + Z_1(N_1 - \lambda D_1)\} , \qquad (4.2)
\end{aligned}$$

$$Z_2 \frac{d\phi_2}{dT} = -\varepsilon\{hZ_1R_{21}\cos(\phi_1 + \phi_2 + \theta) + Z_2(N_2 - \lambda D_2)\} ,$$

where $R_{12} = (U_{12}{}^2 + V_{12}{}^2)^{1/2}$ and $R_{21} = (U_{21}{}^2 + V_{21}{}^2)^{1/2}$. In order to examine the trivial solution of the above equation, Eq. (4.2) is transformed from $\underline{Z}$, $\underline{\phi}$ to new variables $\underline{X}$, $\underline{Y}$ by means of the transformation

$$X_{1,2} = Z_{1,2}\cos\phi_{1,2} \quad , \quad Y_{1,2} = Z_{1,2}\sin\phi_{1,2} \ .$$

This procedure yields a set of nonlinear equations in X and Y as

$$\begin{aligned}
\frac{dX_1}{dT} &= -\varepsilon\{E^*\xi_1X_1 + h(U_{12}Y_2 + V_{12}X_2) + \lambda D_1Y_1 \\
&\quad - [N_{11}(X_1{}^2 + Y_1{}^2) + 2N_{12}(X_2{}^2 + Y_2{}^2)]Y_1\} , \\
\frac{dX_2}{dT} &= -\varepsilon\{E^*\xi_2X_2 + h(U_{21}Y_1 + V_{21}X_1) + \lambda D_2Y_2 \\
&\quad - [2N_{21}(X_1{}^2 + Y_1{}^2) + N_{22}(X_2{}^2 + Y_2{}^2)]Y_2\} , \\
\frac{dY_1}{dT} &= -\varepsilon\{E^*\xi_1Y_1 + h(U_{12}X_2 - V_{12}Y_2) - \lambda D_1X_1 \qquad\qquad (4.3) \\
&\quad + [N_{11}(X_1{}^2 + Y_1{}^2) + 2N_{12}(X_2{}^2 + Y_2{}^2)]X_1\}, \text{ and} \\
\frac{dY_2}{dT} &= -\varepsilon\{E^*\xi_2Y_2 + h(U_{21}X_1 - V_{21}Y_1) - \lambda D_2X_2 \\
&\quad + [2N_{21}(X_1{}^2 + Y_1{}^2) + N_{22}(X_2{}^2 + Y_2{}^2)]X_2\} .
\end{aligned}$$

It is evident that the linear part of these nonlinear equations correspond to the linear variational equations about the trivial solution. Following the procedure given in Appendix C, the stability conditions are obtained as

$$(\xi_1 + \xi_2) > 0,$$

$$|1 - \nu/\omega_o| > [(\frac{\xi_1}{\xi_2})^{1/2} + (\frac{\xi_2}{\xi_1})^{1/2}](h^2R_p - E^{*2}\xi_1\xi_2)^{1/2}/(D_1 + D_2). \quad (4.4)$$

where $R_p = U_{12}U_{21} - V_{12}V_{21}$. Furthermore, the stability conditions reduce to

$$|1 - \nu/\omega_o| > [(\frac{\xi_1}{\xi_2})^{1/2} + (\frac{\xi_2}{\xi_1})^{1/2}]\, hR_p^{1/2} / (D_1 + D_2) \quad (4.5)$$

and

$$|1 - \nu/\omega_o| > 2hR_p^{1/2} / (D_1 + D_2) \quad (4.6)$$

for the cases where the system is lightly damped and undamped, respectively. It is evident from Eqs. (4.4) and (4.5) that R_p has to be positive for the existence of the stability boundary.

One can now consider the case $|\omega_1 - \omega_2| = \omega_o$, again, applying the averaging operator to Eq. (2.21) yields a set of averaged equations given in Appendix A and expresses as

$$\dot{Z}_1 = \varepsilon\{hZ_2[\hat{U}_{12}\sin(\phi_1 - \phi_2) + \hat{V}_{12}\cos(\phi_1 - \phi_2)] + Z_1E^*\xi_1\},$$

$$\dot{Z}_2 = \varepsilon\{hZ_1[\hat{U}_{21}\sin(\phi_1 - \phi_2) + \hat{V}_{21}\cos(\phi_1 - \phi_2)] + Z_2E^*\xi_2\},$$

$$Z_1\dot{\phi}_1 = \varepsilon\{hZ_2[\hat{U}_{12}\cos(\phi_1 - \phi_2) - \hat{V}_{12}\sin(\phi_1 - \phi_2)] + (N_1 - \lambda D_1)Z_1\},$$

$$Z_2\dot{\phi}_2 = \varepsilon\{- hZ_1[\hat{U}_{21}\cos(\phi_1 - \phi_2) - \hat{V}_{21}\sin(\phi_1 - \phi_2)] + (N_1 - \lambda D_r)Z_2\},$$

where the quantities $\hat{U}_{12}$, $\hat{U}_{21}$, $\hat{V}_{12}$ and $\hat{V}_{21}$ are defined in Appendix B-I. As before, it may be shown that the term $(\hat{U}_{12}\hat{V}_{21} - \hat{V}_{12}\hat{U}_{21})$ is identically

zero. Following the same procedure given for the case of $|\omega_1 + \omega_2| = \omega_o$, one obtains the stability conditions as

$$(\xi_1 + \xi_2) > 0 \ ,$$

$$|1 - \nu/\omega_o| > [(\frac{\xi_1}{\xi_2})^{1/2} + (\frac{\xi_2}{\xi_1})^{1/2}]\ [h^2\hat{R}_p - E^*\xi_1\xi_2]^{1/2}/(D_1 + D_2) \qquad (4.7)$$

and

$$|1 - \nu/\omega_o| > 2h\hat{R}_p^{1/2} / (D_1 + D_2) \ , \qquad (4.8)$$

for the cases where the system is damped and undamped, respectively. It is evident form Eqs. (4.7) and (4.8) that the term $\hat{R}_p = \hat{U}_{12}\hat{U}_{21} + \hat{V}_{12}\hat{V}_{21}$, has to be positive for the existence of the stability boundary. However, the numerical calculation indicate that in the primary region, i.e., the region $\bar{u}_o < (\bar{u}_o)_c$, here $(\bar{u}_o)_c$ is the critical mean velocity at which the system loses its stability through divergence, $\hat{R}_p$ in Eqs. (4.7) and (4.8) is negative. It is evident from the stability conditions obtained previously that the system is always stable (no stability boundaries) for $|\omega_1 - \omega_2| = \omega_o$. Thus, in the remaining section of this chapter, we shall only consider the combination resonance of the type $|\omega_1 + \omega_2| = \omega_o$ for $\bar{u}_o < (\bar{u}_o)_c$.

4.2 Stability of the Nontrivial Solution

In order to study the nontrivial solution of Eq. (4.2), the stationary states, apart from the trivial solution $Z_r = 0$, are determined by putting $dZ_r/dT = 0$ and $d\phi_r/dT$ = const. in Eq. (4.2) and from the first two equations of (4.2) yield

$$\frac{Z_{10}}{Z_{20}} = \left(\frac{V_{12}\,\xi_2}{V_{21}\,\xi_1}\right)^{1/2} \tag{4.9a}$$

and

$$\sin(\phi_1 + \phi_2 + \theta) = \frac{-E^*(\xi_1\xi_2)^{1/2}}{hR_p^{1/2}} ,$$

$$\cos(\phi_1 + \phi_2 + \theta) = \pm\left[1 - \frac{E^{*2}\xi_1\xi_2}{h^2 R_p}\right]^{1/2} . \tag{4.9b}$$

By adding the phase equations (4.2), one obtains

$$\lambda(D_1 + D_2) = h\left[\left(\frac{V_{21}\,\xi_1}{V_{12}\,\xi_2}\right)^{1/2} \sqrt{U_{12}^2 + V_{12}^2} + \left(\frac{V_{12}\,\xi_2}{V_{21}\,\xi_1}\right)^{1/2} \sqrt{U_{21}^2 + V_{21}^2}\,\right]$$

$$\cdot \cos(\phi_1 + \phi_2) + (N_1 + N_2) , \tag{4.10}$$

substituting Eq. (4.9) into Eq. (4.10) yields an amplitude-frequency relationship as

$$\lambda = (D_1 + D_2)^{-1}\left\{\pm\left[\left(\frac{\xi_1}{\xi_2}\right)^{1/2} + \left(\frac{\xi_2}{\xi_1}\right)^{1/2}\right]\sqrt{h^2 R_p - E^{*2}\xi_1\xi_2}\right.$$

$$\left. + Z_{10}^2\left[(N_{11} + 2N_{12}) + \left(\frac{V_{21}}{V_{12}}\right)(N_{22} + 2N_{21})\right]\right\} , \tag{4.11}$$

the a-f relationship corresponding to positive and negative signs will be called the positive and negative nontrivial solutions.

The stability of the nontrivial bifurcating solution is examined by considering the linear variational equation of the averaged Eq. (4.2) about the nontrivial solution. Letting $Z_1 = Z_{10} + X_1$, $Z_2 = Z_{20} + X_2$ and $\phi = \phi_o + Y$ where $\phi_o = \phi_{10} + \phi_{20} + \theta$, the linear variational equations can

be reduced to

$$\frac{dX_1}{dT} = -\varepsilon\{E^*\xi_1 X_1 - E^*\xi_1 \frac{Z_{20}}{Z_{10}} X_2 + hZ_{20}R_{12}\cos\phi_o Y\} ,$$

$$\frac{dX_2}{dT} = -\varepsilon\{- E^*\xi_2 \frac{Z_{10}}{Z_{20}} X_1 + E^*\xi_2 X_2 + hZ_{10}R_{21}\cos\phi_o Y\}, \qquad (4.12)$$

$$\frac{dY}{dT} = -\varepsilon\{[\mp \frac{1}{Z_{10}} F\alpha^{1/2} + (\frac{\partial N_1}{\partial Z_1} + \frac{\partial N_2}{\partial Z_1})_{Z_{10},Z_{20}}] X_1$$

$$+ [\pm \frac{1}{Z_{20}} F\alpha^{1/2} + (\frac{\partial N_1}{\partial Z_2} + \frac{\partial N_2}{\partial Z_2})_{Z_{10},Z_{20}}] X_2 + E^*(\xi_1 + \xi_2)Y\} ,$$

where

$$F = [(\xi_1/\xi_2)^{1/2} - (\xi_2/\xi_1)^{1/2}] , \quad \alpha = h^2 R_p - E^{*2}\xi_1\xi_2 , \quad \frac{Z_{10}}{Z_{20}} = (\frac{V_{12}\xi_2}{V_{21}\xi_1})^{1/2} .$$

In Eqs. (4.12), Z_{ro} and ϕ_{ro}, $r = 1,2$ are stationary nontrivial solutions. Thus, using the a-ſ relation derived from Eq. (4.11)

$$\nu = \omega_o \mp [(\xi_1/\xi_2)^{1/2} + (\xi_2/\xi_1)^{1/2}]\, \alpha^{1/2} - Z_{10}^2[N_{11} + 2N_{21}$$

$$+ (\frac{V_{21}\xi_1}{V_{12}\xi_2})(N_{22} + 2N_{12})] , \qquad (4.13)$$

the characteristic equation for Eqs. (4.12) can be obtained as

$$\rho^3 + 2E^*(\xi_1 + \xi_2)\rho^2 + [E^{*2}(\xi_1 + \xi_2)^2 + F^2\alpha \mp \alpha^{1/2}[\sigma_1 Z_{10}(\xi_1/\xi_2)^{1/2}$$

$$+ \sigma_2 Z_{20}(\xi_2/\xi_1)^{1/2}]\rho \mp 2E^*(\xi_1\xi_2)^{1/2}\alpha^{1/2}[Z_{10}\sigma_1 + Z_{20}\sigma_2] = 0 , \qquad (4.14)$$

where

$$\sigma_1 = \frac{\partial N_1}{\partial Z_1} + \frac{\partial N_2}{\partial Z_1} = 2(N_{11} + 2N_{21})Z_{10} ,$$

$$\sigma_2 = \frac{\partial N_1}{\partial Z_2} + \frac{\partial N_2}{\partial Z_2} = 2(N_{22} + 2N_{12})Z_{20} .$$

For the cubic Eq. (4.14), the condition that none of the roots has positive real parts is given by the Routh-Hurwitz criteria, which requires that

$$(\xi_1 + \xi_2) > 0 ,$$

$$E^{*2}(\xi_1 + \xi_2)^2 + F^2\alpha \mp 2\alpha^{1/2}(\xi_1/\xi_2)^{1/2}Z_{10}^2[(N_{11} + 2N_{12})$$
$$+ (\frac{V_{21}}{V_{12}}) (N_{22} + 2N_{21})] > 0,$$

$$\mp 2E^*(\xi_1\xi_2)^{1/2}\alpha^{1/2}Z_{10}^2[(N_{11} + 2N_{12}) + (\frac{\xi_1 V_{21}}{\xi_2 V_{12}}) (N_{22} + 2N_{21})] > 0 ,$$

$$2E^{*3}(\xi_1 + \xi_2)^3 + 2F^2\alpha E^*(\xi_1 + \xi_2) \mp 4E^* \frac{\alpha^{1/2}}{(\xi_1\xi_2)} Z_{10}^2[\xi_1^2(N_{11} + 2N_{12}) \tag{4.15}$$
$$+ (\frac{\xi_1 V_{21}}{\xi_2 V_{12}}) \xi_2^2 (2N_{12} + N_{22}) > 0 .$$

Since it can be shown that ξ_1 is positive, it is evident from the stability conditions that for

$$\mp \{N_{11} + 2N_{12} + (\frac{\xi_1}{\xi_2} \frac{V_{21}}{V_{12}}) (N_{22} + 2N_{21})\} = \mp \psi_1 > 0 ,$$

the nontrivial solution corresponding to the positive sign in Eq. (4.13) is stable only if

$$E^{*2}(\xi_1 + \xi_2)^2 + F^2\alpha \mp 2\alpha^{1/2}(\xi_1/\xi_2)^{1/2} Z_{10}^2[(N_{11} + 2N_{12})$$

$$+ (\frac{V_{21}}{V_{12}}) (N_{22} + 2N_{21})] > 0$$

and

$$2[(\xi_1/\xi_2)^{1/2} + (\xi_2/\xi_1)^{1/2}] \cdot [E^{*2}(\xi_1 + \xi_2)^2 + F^2\alpha]$$

$$\mp 4 \frac{\alpha^{1/2}}{(\xi_1\xi_2)} Z_{10}^2[\xi_1^2(N_{11} + 2N_{12}) + (\frac{\xi_1 V_{21}}{\xi_2 V_{12}}) \xi_2^2(2N_{12} + N_{22})] > 0 \ . \qquad (4.16)$$

It should be noted that in the above equation the negative and positive signs correspond to positive and negative nontrivial solutions, respectively. Furthermore, for light damping, i.e., $E^* \to 0$ and $\xi_1 \approx \xi_2$, the linear variational equation of the averaged Eq. (4.2) can be obtained as

$$\frac{dX_1}{dT} = - \epsilon h Z_{20} R_{12} \cos\phi_o Y \ ,$$

$$\frac{dX_2}{dT} = - \epsilon h Z_{10} R_{21} \cos\phi_o Y \ ,$$

$$\frac{dY}{dT} = - \epsilon\{[\mp \frac{1}{Z_{10}} FhR_p^{1/2} + (\frac{\partial N_1}{\partial Z_1} + \frac{\partial N_2}{\partial Z_2})_{Z_{10},Z_{20}}] X_1$$

$$+ [\pm \frac{1}{Z_{20}} FhR_p^{1/2} + (\frac{\partial N_1}{\partial Z_2} + \frac{\partial N_2}{\partial Z_1})_{Z_{10},Z_{20}}] X_2 \ ,$$

the amplitude-frequency relation becomes

$$\nu = \omega_o \mp hR_p^{1/2} [(\xi_1/\xi_2)^2 + (\xi_2/\xi_1)^2] - Z_{10}^2[(N_{11} + 2N_{21})$$

$$+ (\frac{V_{21}\xi_1}{V_{12}\xi_2}) (N_{22} + 2N_{12})] , \qquad (4.17)$$

and the corresponding stability condition is

$$F^2hR_p^{1/2} \mp 2(\xi_1/\xi_2)^{1/2} Z_{10}^2[(N_{11} + 2N_{12}) + (\frac{V_{21}}{V_{12}}) (N_{22} + 2N_{21})] > 0 . \qquad (4.13)$$

Thus, for $[(N_{11} + 2N_{12}) + (V_{21}/V_{12}) (N_{22} + 2N_{21})] = \psi_2 < 0$, the nontrivial solution corresponding to the negative sign is always stable while that corresponding to the positive sign in Eq. (4.17) is stable only if

$$Z_{10} < (\xi_2/\xi_1)^{1/2} \frac{F^2hR_p^{1/2}}{2[(N_{11} + 2N_{12}) + (V_{21}/V_{12})(N_{22} + 2N_{21})]} . \qquad (4.19)$$

Evidently, opposite results prevail for $\psi_2 > 0$. The analysis for the undamped system can be carried out be letting $\xi_1 = \xi_2$ in Eq. (4.17). Thus, for the undamped system, the a-f relation is obtained as

$$\nu = \omega_o \mp 2hR_p^{1/2} - Z_{10}^2[N_{11} + 2N_{12}) + (\frac{V_{21}}{V_{12}}) (N_{22} + 2N_{21})] \qquad (4.20)$$

Since F = 0 for the undamped system, by examining the condition (4.18), one can infer that for $\psi_2 < 0$, the undamped nontrivial solution corresponding to the negative sign is always stable while that correspoinding to the positive sign is always unstable. Furthermore, for $\psi_2 > 0$, the opposite results prevail.

4.3 Bifurcation Analysis of Trivial Solution

As pointed out in Appendix C, the trivial solution of Eq. (4.4) losses stability at $\lambda = \pm\lambda_c$ (i.e., $\lambda_c = \pm\,[(\xi_1/\xi_2)^{1/2} + (\xi_2/\xi_1)^{1/2}]\eta^{1/2} / (D_1 + D_2))$, and undergoes a Hopf bifurcation. In order to examine the local bifurcation behavior, the linear part of these equations is brought to the simplest diagonal form with the help of the transformation $(\underline{X},\underline{Y} \rightarrow \underline{u},\underline{v})$ given in Appendix C. This procedure yields

$$\dot{\underline{u}} = B(\lambda_c^{\pm})\,\underline{u} + g(\underline{u},\underline{v},\lambda_{c,\eta}^{\pm})\ ,\quad \dot{\underline{v}} = C(\lambda_c^{\pm})\underline{v} + h(\underline{u},\underline{v},\lambda_{c,\eta}^{\pm})\ ,$$

where $B(\lambda_c)$ and $C(\lambda_c)$ are defined in Appendix C by Eq. (C6). It may be noted that the nonlinear functions g and h are different for λ_c^{+} and λ_c^{-} due to the fact that the *eigenvectors of* B and C at λ_c^{+} and λ_c^{-} are different even though the eigenvalues are the same. For the problem under consideration, the nonlinear terms are cubic in $\underline{u}$ and $\underline{v}$. Thus, the contribution from the stable modes $\underline{v}$ to the equations restricted to the center manifold is of the order $|u|^k$, $k > 3$, and can be neglected in the first approximation [28]. The bifurcating periodic solutions are given [28] near $\lambda = \pm\lambda_c$ as

$$u_1 = \hat{z}\sin\hat{\psi}\ ,\quad u_2 = \hat{z}\cos\hat{\psi}\ , \tag{4.21}$$

where

$$\hat{z}^2 = -\left(\delta_1'\gamma/R\right)_{\lambda=\lambda_c^{\pm}}\ ,\quad \psi = \bar{\omega}t + \bar{\alpha}\ ,$$

$$\bar{\omega} = \left\{\omega_1 + \gamma\left(\omega_1' - \frac{S}{R}\,\delta_1'\right)\right\}_{\lambda=\lambda_c^{\pm}}\ ,$$

$$\gamma = (\nu/\omega_o) - (\nu/\omega_o)_c^{\pm}\ ,$$

$$(\nu/\omega_o)_c^{\pm} = 1 \mp \left[\left(\frac{\xi_1}{\xi_2}\right)^{1/2} + \left(\frac{\xi_2}{\xi_1}\right)^{1/2}\right]\eta^{1/2} \,/\, (D_1 + D_2) \,.$$

$$R = \frac{1}{16}\left\{\frac{\partial^3 g_1}{\partial u_1^{\,3}} + \frac{\partial^3 g_1}{\partial u_1 \partial u_2^{\,2}} + \frac{\partial^3 g_2}{\partial u_1^{\,2}\partial u_2} + \frac{\partial^3 g_2}{\partial u_2^{\,3}}\right\}_{\underset{\sim}{u} = \underset{\sim}{v} = 0} \,, \tag{4.22}$$

$$S = \frac{1}{16}\left\{\frac{\partial^3 g_1}{\partial u_2^{\,3}} + \frac{\partial^3 g_1}{\partial u_1^{2}\partial u_2} - \frac{\partial^3 g_2}{\partial u_1 \partial u_2^{\,2}} - \frac{\partial^3 g_2}{\partial u_1^{\,3}}\right\}_{\underset{\sim}{u} = \underset{\sim}{v} = 0} \,.$$

The expressions for R and S are obtained from g_1 and g_2, which is given in Appendix C by Eq. (C8). In addition, it can be shown that

$$Z_{ro} = 4[(c_r^1)^2 + (d_r^1)^2]\hat{z}^2 \,, \quad r = 1,2 \,, \tag{4.23}$$

where Z_{ro} and $\hat{z}^2$ are defined in Eqs. (4.13) and (4.21) respectively. Furthermore, at $(\nu/\omega_o) = (\nu/\omega_o)_c^{\pm}$ the derivative of the real part of the critical eigenvalue, δ_1', is negative and positive respectively. Thus, the nontrivial solutions exist for $\Upsilon > 0$ only if $R < 0$ and $R > 0$ at λ_c^+ and λ_c^- respectively. Furthermore, the stability of the trivial and bifurcation paths are given by

$$\frac{dw}{dt} = \delta'\Upsilon w \quad \text{and} \quad \frac{dw}{dt} = -2\delta'\Upsilon w \,. \tag{4.24}$$

Since in this problem $R < 0$ at λ_c^+ and $R > 0$ at λ_c^- (from numerical calculations), it is obvious that at λ_c^+ the averaged system undergoes a stable supercritical Hopf bifurcation and at λ_c^- the averaged system undergoes an unstable subcritical Hopf bifurcation. The undamped system

becomes unstable at λ_c given by Eq. (4.4), due to the eigenvalues in the imaginary axis coalescing and one pair crossing to the left and the other to the right half of the complex λ-plane. Such instabilities in the context of Hamiltonian systems give rise to a so-called Hamiltonian Hopf bifurcation [29]. The linear operator at this critical λ value has a non-semisimple form, and the bifurcation behavior is yet to be examined in detail.

NUMERICAL METHOD

5.1 Numerical Results of Analytical Method

The numerical results of pinned-pinned pipe and clamped-clamped pipe for subharmonic and combination resonance cases are discussed here. In order to investigate the parametric instability, the unperturbed system is assumed stable and only the primary stable region will be considered, i.e., $\bar{u}_o < (\bar{u}_o)_c$ where $(\bar{u}_o)_c$ is π and 2π for pinned-pinned and clamped-clamped pipes, respectively [17] and represents the critical mean velocity at which the system become unstable through divergence.

We first discuss the case of subharmonic resonance. The numerical calculations indicate that, in the primary stable region the terms ξ_r, $U_{rr}^2 + V_{rr}^2$ appearing in Eqs. (3.1) are positive. Thus, it is evident from the stability conditions (3.6), that the stability boundaries exist for $\nu = 2\omega_1$, $\nu = 2\omega_2$. In Figs. 1a and 2a the relationships between μ and ν/ω_{o1} (ω_{o1} is the dimensionless frequency in the first mode when $\bar{u}_o = 0$) for pinned-pinned pipe with $\epsilon E^* = 0.015$ and $\epsilon E^* = 0.005$ are shown for the cases $\nu = 2\omega_1$ and $\nu = 2\omega_2$, respectively for $\bar{u}_o = 1.88$. Similarly, the stability boundaries for a clamped-clamped pipe with $\epsilon E^* = 0.005$ and $\epsilon E^* = 0.001$ are shown in Figs. 3a and 4a for the cases $\nu = 2\omega_1$ and $\nu = 2\omega_2$, respectively. In Figs. 1a - 4a the points S and D_S represent the parameter values at which the instability of the trivial solution of the averaged system Eqs. (3.1) occurs through a simple bifurcation (one eigenvalue crossing the origin in the complex λ-plane) and double zero bifurcation (two zero eigenvalues crossing the origin), respectively. The associated bifurcation paths are shown in Figs. 1b - 4b. For a fixed value of μ, as ν is increased, the trivial solution loses stability at left point

D_S or S depending on whether the system is damped or undamped. The nontrivial solution defined by Eq. (3.4) bifurcates at either point D_S or S, giving a stable nonzero but constant Z_{ro} value. The corresponding solutions of Eq. (2.21) are periodic with period $2\pi/\omega_r$, $(\omega_r = \nu/2)$, where ω_r and Z_r are related by the relationship (Eq. (3.4)) as shown in Figs. 1b - 4b. On the other hand, if ν is decreased from the right to left, the trivial solution becomes unstable at right point D_S or S and bifurcating solution is unstable as shown in Figs. 1b - 4b.

Secondly, for the case of combination resonance. The numerical calculations indicate that in the primary stable region the terms ξ_1, ξ_2 and $U_{12}U_{21} + V_{12}V_{21}$ appearing in Eq. (4.1) are positive. The stability boundaries for pinned-pinned pipe with $\epsilon E^* = 0.005$ for $\bar{u}_o = 1.88$ and clamped-clamped pipe with $\epsilon E^* = 0.001$ for $\bar{u}_o = 4.0$ are given in Figs. 5a and 6a, respectively. The points H and D_H in Figs. 5a and 6a represent the parameter values at which the instability of the trivial solution of the averaged system (4.2) occurs through a Hopf bifurcation (a pair of conjugate eigenvalues crossing the imaginary axis in the complex λ-plane) and double Hopf bifurcation (defined before as a Hamiltonian Hopf bifurcation in Hamiltonian system), respectively. The associated bifurcation paths are shown in Figs. 5b and 6b. For a fixed value of μ as ν is increased, the trivial solution loses stability at the left point H and stable supercritical Hopf bifurcation takes place and a periodic path branches off. The corresponding solutions of Eq. (4.2) are modulated periodic solutions. On the other hand, at the right point H, the trivial solution loses stability, as ν is decreased from far right, through an unstable subcritical Hopf bifurcation.

The numerical results of the local bifurcation analysis pertaining to Eqs. (4.21) -(4.24) are calculated for pinned-pinned and clamped-clamped pipes. Even though the numerical results differ from that of the previous results in the third digit, the plots of these two results are nearly identical as shown in Figs. 7 and 8 for pinned-pinned and clamped-clamped pipe, respectively. It is evident from the numerical calculation that the value of R and δ' in Eq. (4.22) is $\mp$ 0.00997 and $\mp$ 0.0616 respectively, for pinned-pinned condition at $\lambda_c^{\pm}$. On the other hand, R and δ' are $\mp$ 0.00546 and $\mp$ 0.131 respectively, for clamped-clamped condition at $\lambda_c^{\pm}$.

5.2 Numerical Scheme for Periodic Solutions of Autonomous System

The analytical results, of the periodic solutions of the averaged equations, obtained using the Hopf bifurcation theorem is valid only in the small neighborhood of $\lambda = \lambda_c^{\pm}$. For large values of $\gamma = \lambda - \lambda_c^{\pm}$, these solutions become inaccurate and therefore it is necessary to use a numerical scheme which can determine the bifurcating periodic solutions of the autonomous averaged Eqs. (4.3). The straightforward method of calculating steady-state periodic solutions of nonlinear autonomous Eqs. (4.3)

$$\dot{\underset{\sim}{a}} = \underset{\sim}{f}(\underset{\sim}{a},\mu,\lambda)\ ,\ \underset{\sim}{a}(o) = \underset{\sim}{a}_o,\ \ \underset{\sim}{a} = (\underset{\sim}{X},\underset{\sim}{Y})\ , \tag{5.1}$$

is to numerically integrate the differential equations from some initial state until the transient response becomes negligible. However, in a lightly damped system, convergence to the steady-state response is very slow, and the integration must extend over many periods making the computation time consuming and costly. Thus, in this study, we shall make use of Newton-Raphson algorithm given by Aprille and Trick [30] to

determine the period T of the response and the initial condition $\underline{a}_o$ such that integrating (5.1) from the guessing initial condition $\underline{a}_o$ over the time interval [0,T] yields immediately the steady-state periodic solution of period T. In this method, the initial value problem (5.1) is transformed into a two points boundary value problem

$$\underline{a} = E(T,\underline{a}_o) \ , \quad \text{where } E(T,\underline{a}_o) = \underline{a}_o + \int_0^T \underline{f}(\underline{a}(t),\mu,\lambda)dt, \ T > 0 \qquad (5.2)$$

with 4 equations and 5 unknowns, $\underline{a}_o$, T. The resulting set of nonlinear algebraic equations are then solved using a Newton-Raphson interaction technique. Such algorithms have been used by Tousi and Bajaj [31] to solve periodic solutions of averaged equations.

It is important that the initial guess of the period T and the initial condition $\underline{a}_o$ is reasonably good for fast convergence of this algorithm. The known analytical solutions of the Hopf bifurcation theorem can be used as a good starting point near $\lambda = \lambda_c^{\pm}$. Once the period and the initial conditions are obtained for a specific value of λ, these values can be used as the starting point for calculating the period and initial conditions for $\lambda + \Delta\lambda$, provided $\Delta\lambda$ is sufficiently small. The stability of the periodic solution $a = \phi(\omega t)$ is governed by the variational equation

$$\frac{d\xi}{dt} = A(t)\xi \ , \ A(t) = \frac{\partial f}{\partial a}\left(\phi(\tau)\right) \ , \ \tau = \omega t \qquad (5.3)$$

whose solution can be written, for example, as $\xi = C_o\phi'(\tau)$ and C_o is an arbitrary constant. The Floquet multipliers are eigenvalues, ζ_i, of the monodromy matrix and one of the eigenvalues is +1. The periodic solutions are stable provided the remaining multipliers lie inside the unit circle.

There are three ways in which the periodic solutions can become unstable: (i) by one multiplier of Eq. (5.3) leaving the unit circle through +1 giving rise to a saddle node bifurcation, (ii) by one multiplier crossing the unit circle at −1 and the associated bifurcation is referred to as a period doubling bifurcation, and (iii) by a pair of complex conjugate multipliers $\zeta, \bar{\zeta}$, with $|\zeta| = 1$, giving rise to a solution on a two-torus T^2. It will be seen in the numerical examples that only saddle node bifurcations take place for the problem under consideration.

The periodic solution obtained using the numerical scheme is shown in Figs. 9 and 10 for pinned-pinned pipe and clamped-clamped pipe, respectively. Even though the numerical method involves a considerable amount of computations, the known analytical solutions were used to reduce this effort significantly. In Figs. 9a and 10a, the amplitude frequency relationships obtained analytically and numerically are compared for the damped system. The period T of the bifurcating orbit is plotted against the frequency $\nu = \omega_1 + \omega_2$ for the damped system in Figs. 9b and 10b.

CONCLUSIONS

In this study, the ideas related to the method of averaging, Poincare-Birkoff normal form [27], and center manifold theorem [28] have been used appropriately at different stages of the analysis to investigate the stability and bifurcation behavior of nonlinear supported pipes conveying pulsating fluid. Explicit results for the stability boundaries of the trivial solution, bifurcating paths and their stability have been obtained for values of the system parameters μ, E^* and ν, where the value of ν is taken in the neighborhood of $\nu = 2\omega_1$, $\nu = 2\omega_2$ and $\nu = \omega_1 + \omega_2$. It is shown that when the system undergoes a combination resonance, the bifurcating solutions of the averaged system are periodic as opposed to constant solutions which bifurcate in the case of subharmonic resonance. Thus, in the case of combination resonance, the original system exhibits a modulated periodic solution or a T^2 solution as opposed to period two solutions in the case of subharmonic resonance.

There are two types of bifurcating paths obtained from the analytical method, namely the "global" and the "local" bifurcation solutions. The "global" bifurcation solutions are obtained directly from the averaged equations as nontrivial solutions. Whereas the "local" bifurcation solutions are obtained by examining the instability of the trivial solution of the averaged equations and its various bifurcations. These two results agree in their common regions of validity. Finally, the numerical scheme which determines the bifurcating periodic solutions is only used for the case $\nu = \omega_1 + \omega_2$. It is evident from the sensitivity of the averaged, autonomous equation that the step of each integration should be sufficiently small.

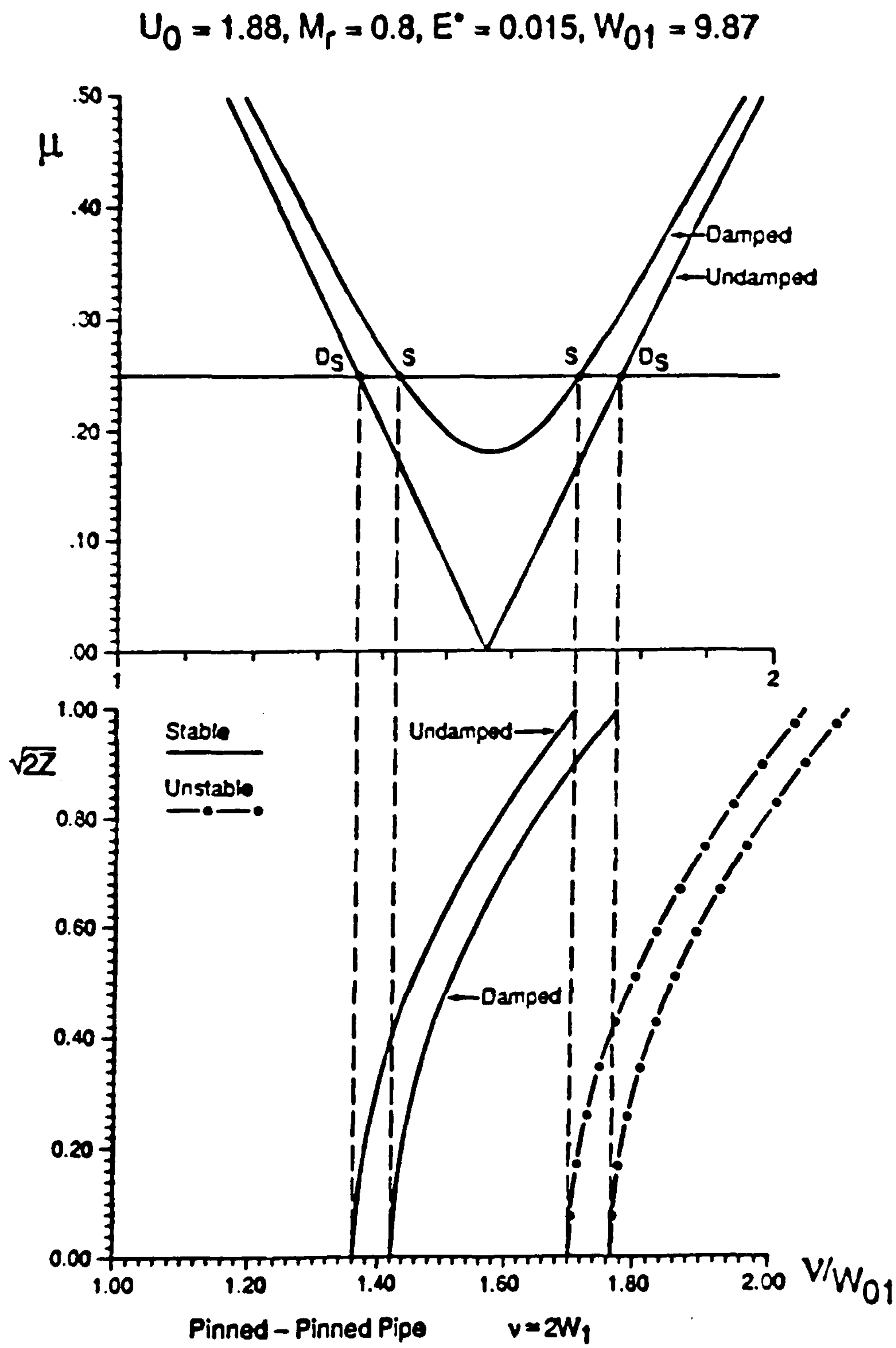


Figure 1 (a) Stability boundaries, and (b) amplitude-frequency relationships for pinned-pinned pipe - subharmonic resonance $\nu = 2\omega_1$, $\omega_1 = 7.71$.

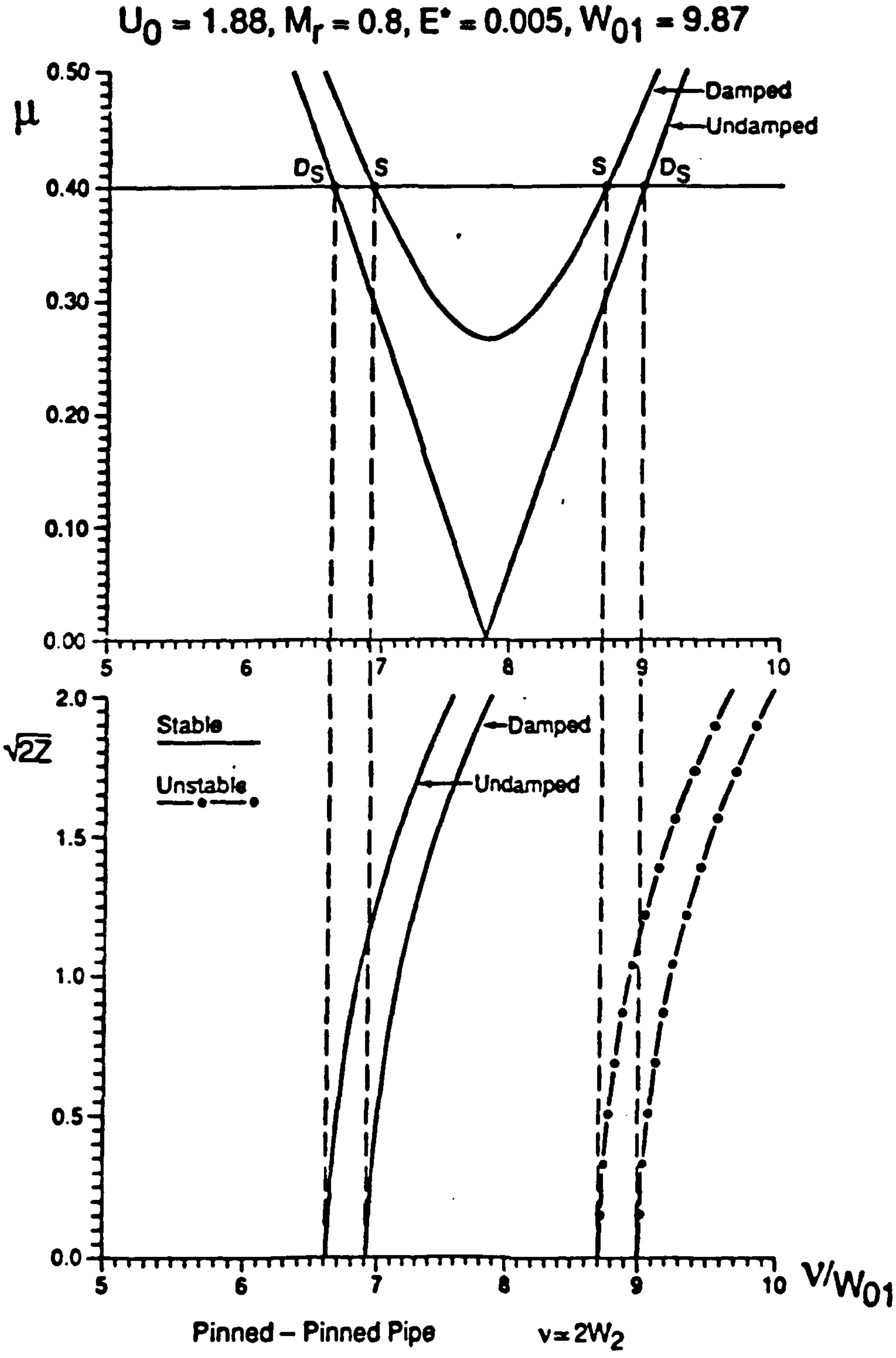


Figure 2 (a) Stability boundaries, and (b) amplitude-frequency relationships for pinned-pinned pipe - subharmonic resonance $\nu = 2\omega_2$, $\omega_2 = 38.55$.

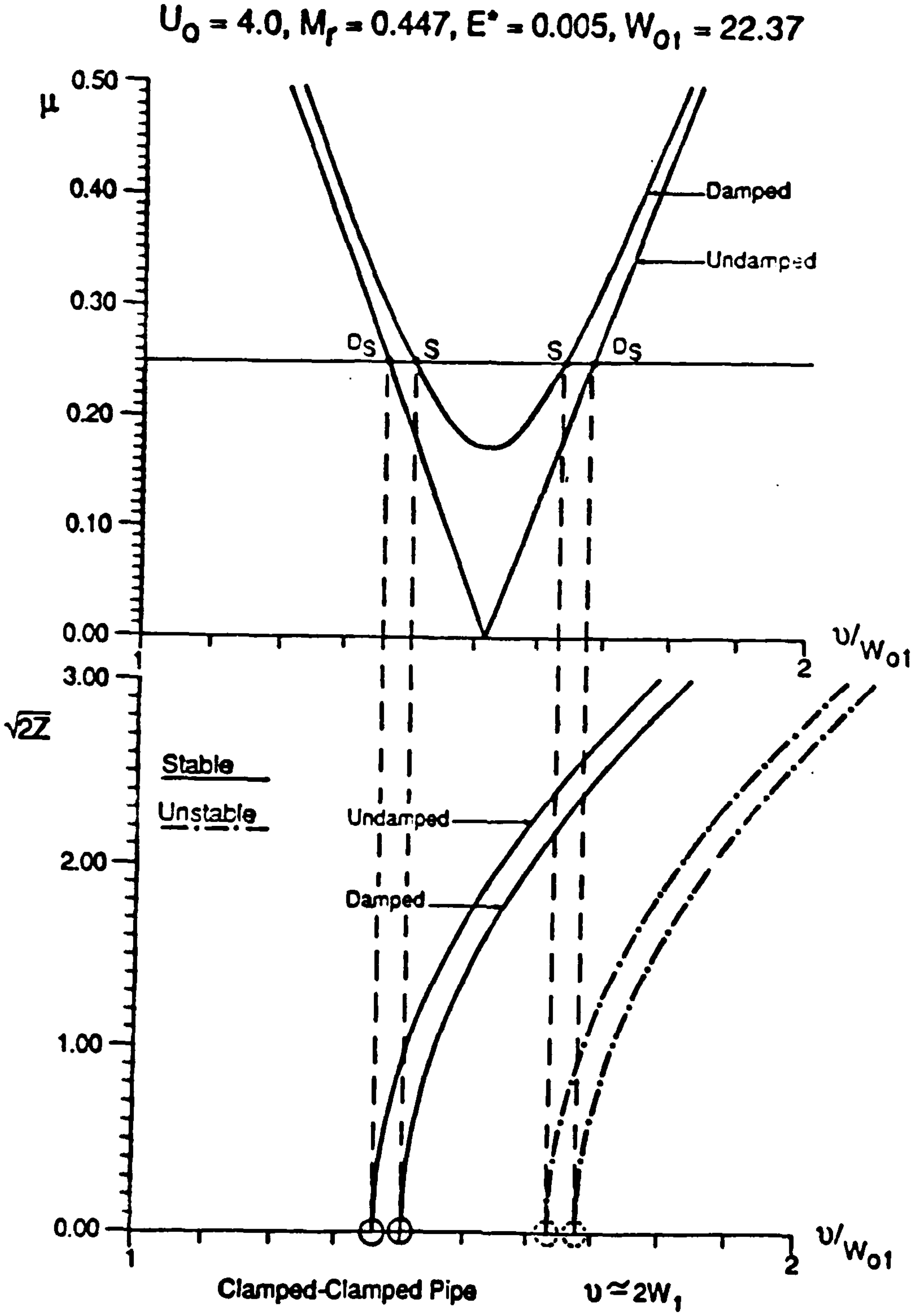


Figure 3 (a) Stability boundaries, and (b) amplitude-frequency relationships for clamped-clamped pipe - subharmonic resonance $\nu = 2\omega_1$, $\omega_1 = 16.98$.

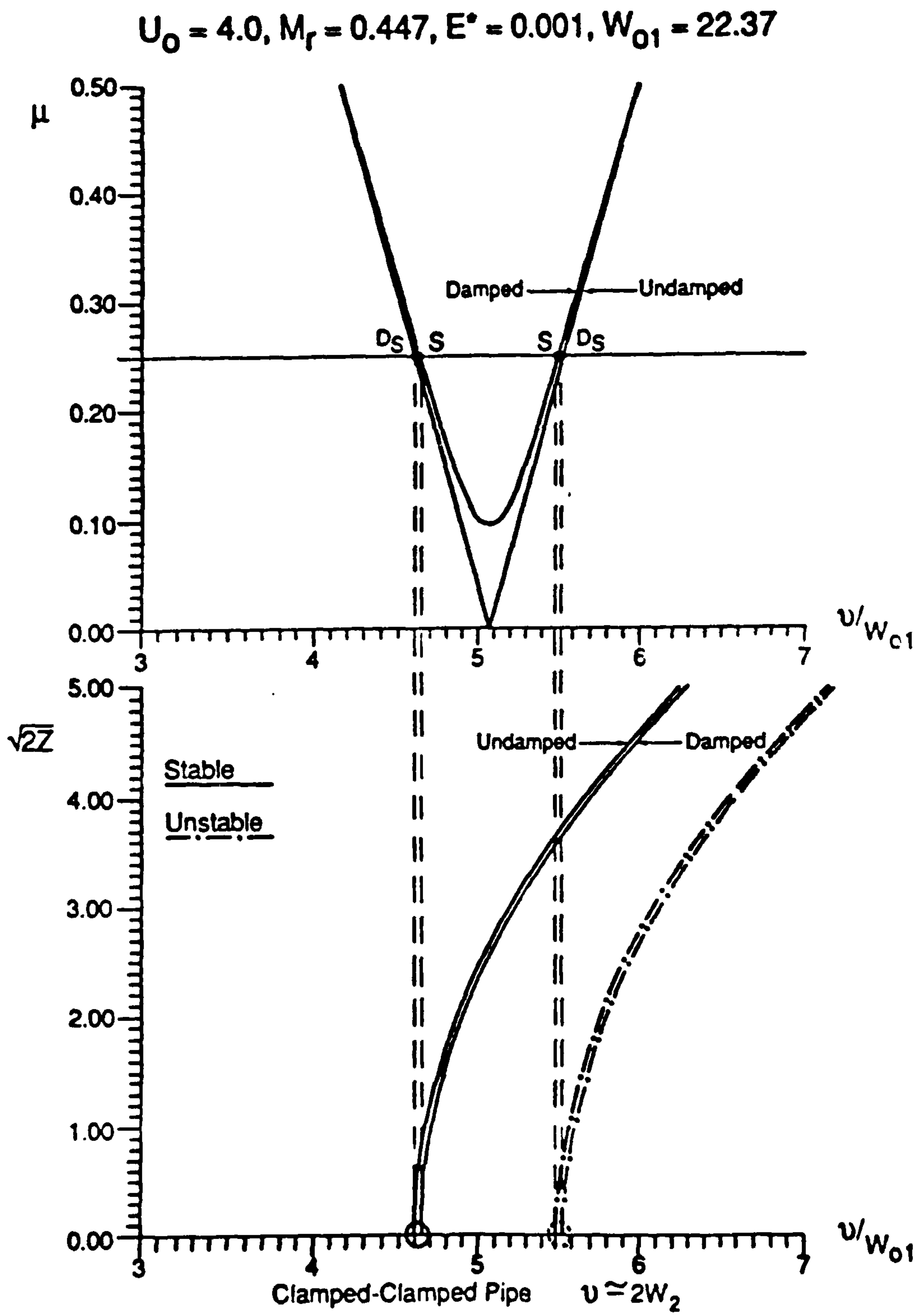


Figure 4 (a) Stability boundaries, and (b) amplitude-frequency relationships for clamped-clamped pipe - subharmonic resonance $\nu = 2\omega_2$, $\omega_2 = 56.77$.

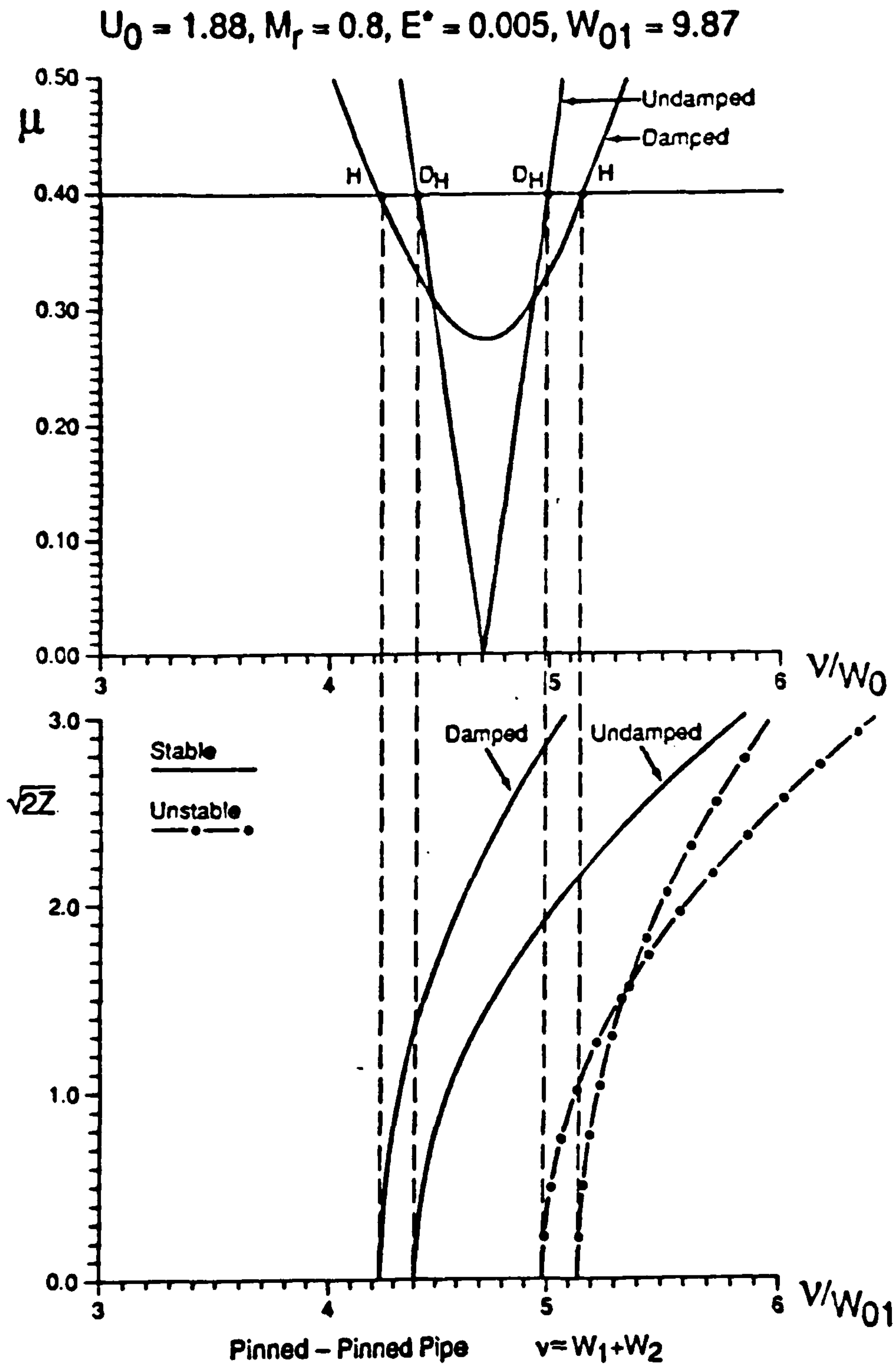


Figure 5 (a) Stability boundaries, and (b) amplitude-frequency relationships for pinned-pinned pipe - combination resonance $\nu = \omega_1 + \omega_2$, $\omega_1 = 7.71$, $\omega_2 = 38.55$.

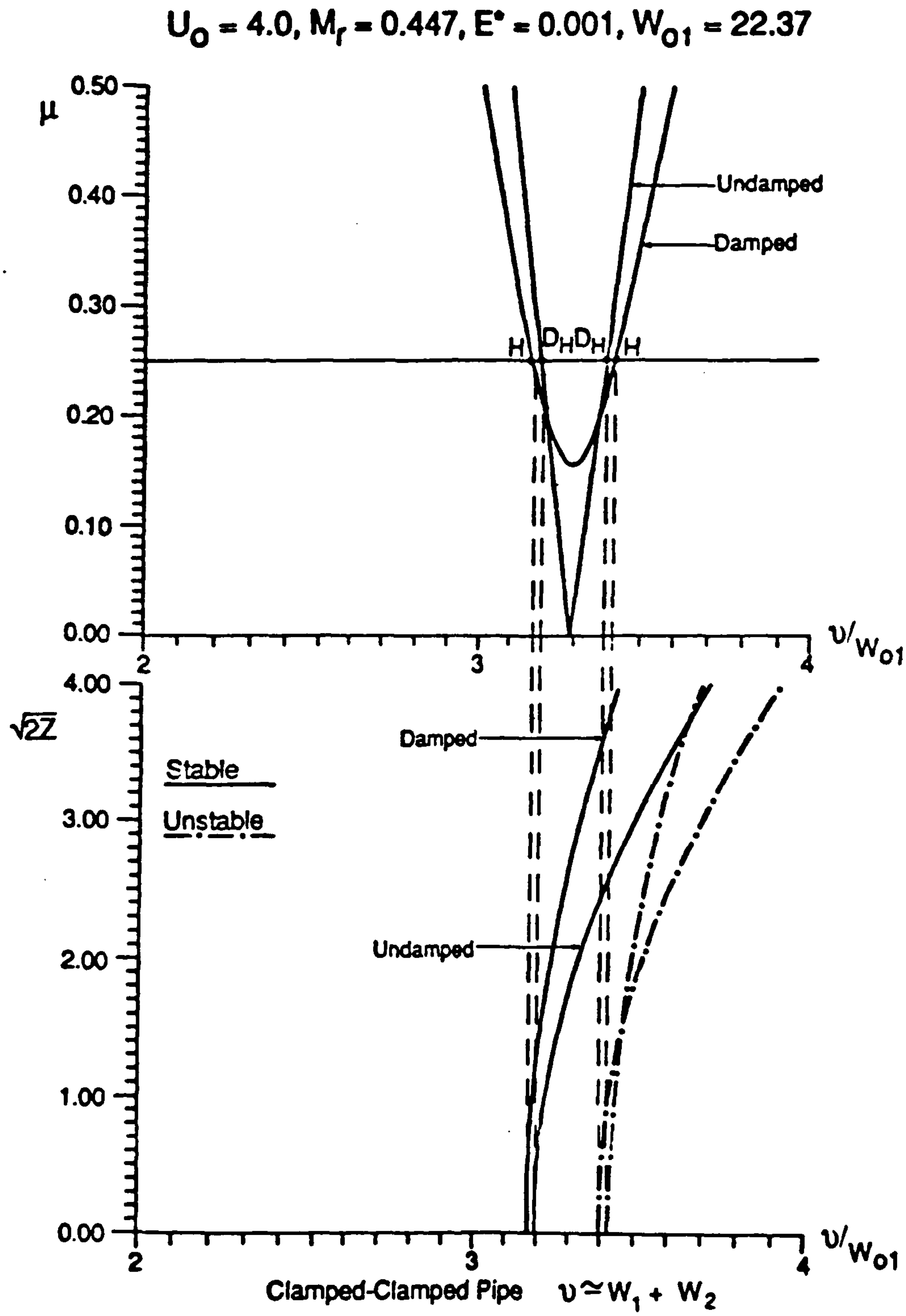


Figure 6 (a) Stability boundaries, and (b) amplitude-frequency relationships for clamped-clamped pipe - combination resonance $\nu = \omega_1 + \omega_2$, $\omega_1 = 16.93$, $\omega_2 = 5$[illegible]$.77$.

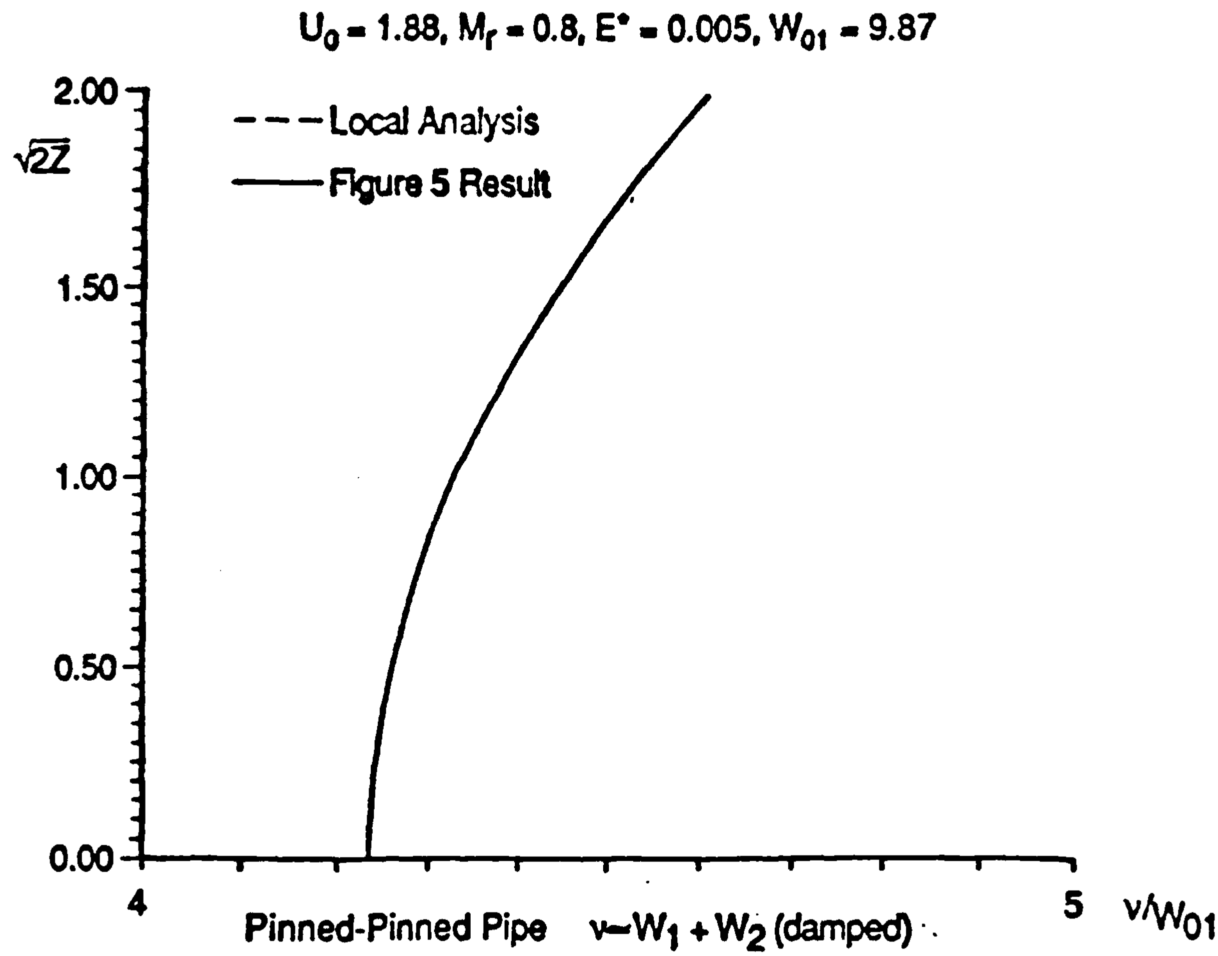


Figure 7 Comparison between local analysis and Figure 5 result for the amplitude frequency relation for pinned-pinned pipe - combination resonance $\nu = \omega_1 + \omega_2$, $\omega_1 = 7.71$, $\omega_2 = 38.55$.

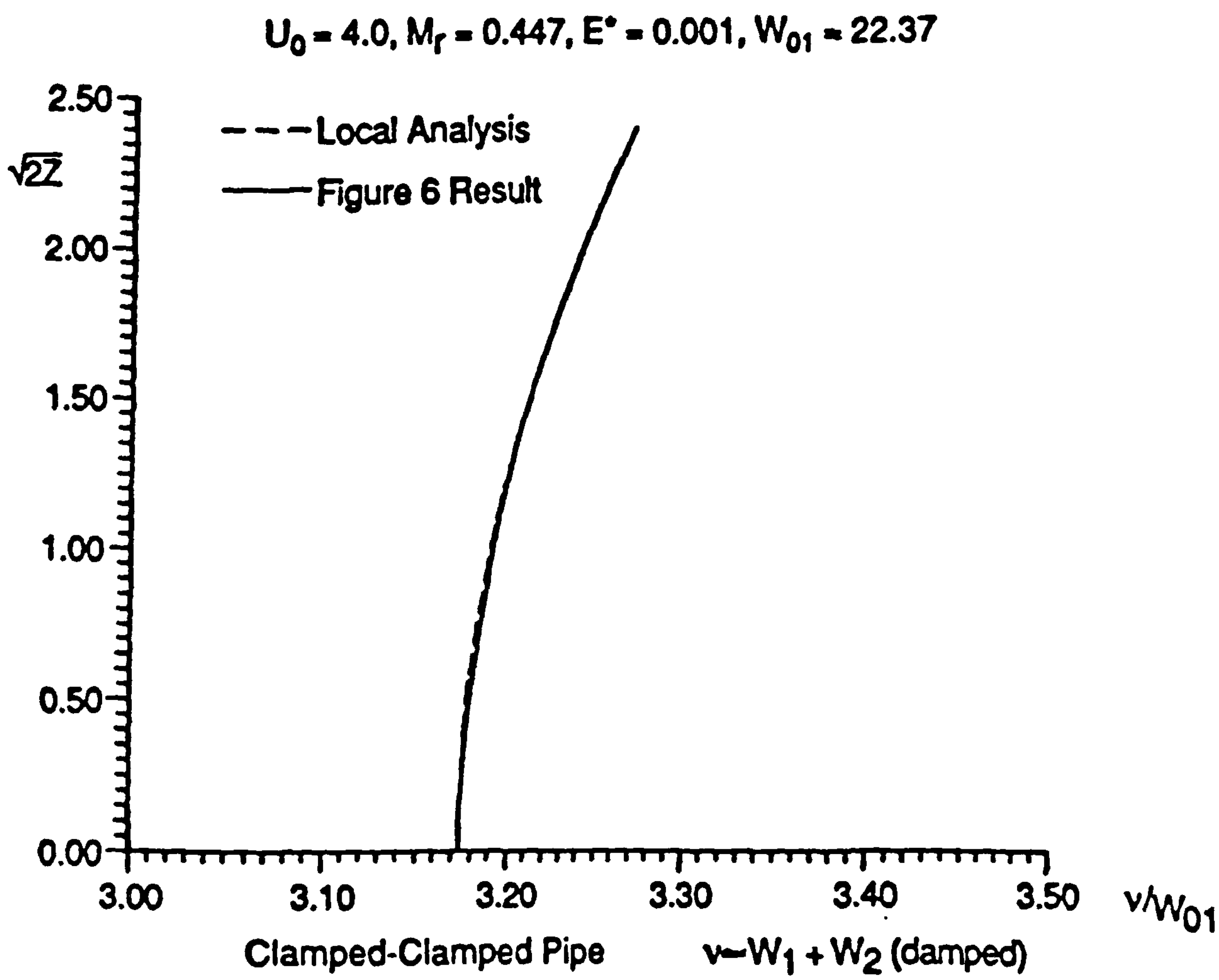


Figure 8 Comparison between local analysis and Figure 6 result for the amplitude frequency relation for clamped-clamped pipe - combination resonance $\nu = \omega_1 + \omega_2$, $\omega_1 = 16.98$, $\omega_2 = 56.77$.

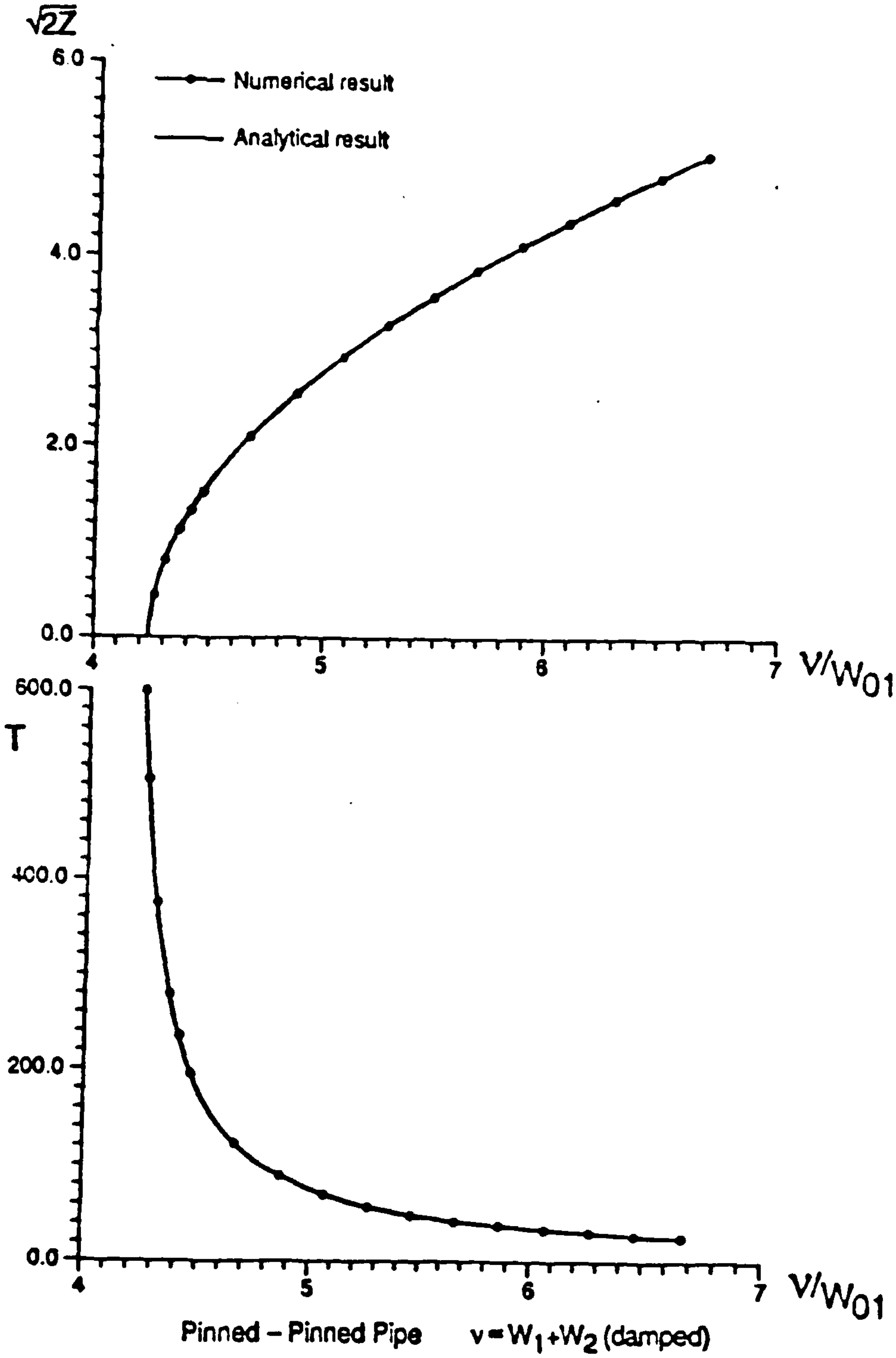


Figure 9 (a) Comparison between analytical and numerical results for the amplitude frequency relation, (b) relationship between period and freqeuncy ν, for pinned-pinned pipe.

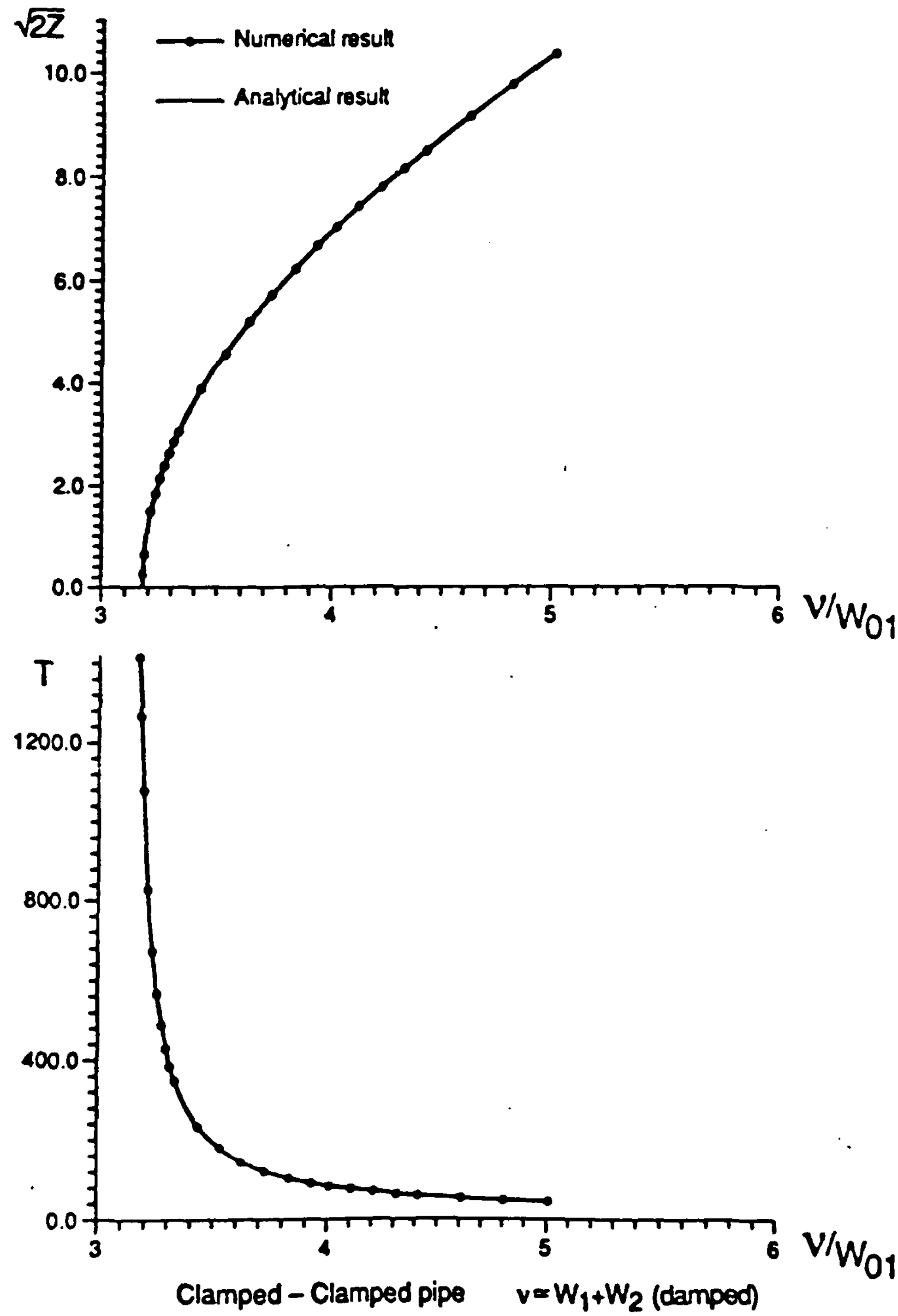


Figure 10 (a) Comparison between analytical and numerical results for the amplitude frequency relation, (b) relationship between period and frequency ν, for clamped-clamped pipe.

APPENDIX F1

AVERAGED EQUATION OF PARAMETRIC EXCITATION

The averaged equation in the first approximation for the following cases of Eq. (2.21) are

(1) $2\omega_1 = \omega_0$

$$\dot{Z}_1 = Z_1\{\mu[\nu D_1(2,2)(\omega_2\alpha_1^2 - \omega_1\alpha_1\alpha_2)$$

$$+ \nu D_1(1,1)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2^2) + D_2(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2)$$

$$+ D_2(1,2)(\omega_2\alpha_1\alpha_2^2 - \omega_1\alpha_1^2\alpha_2) + D_3(2,2)(\omega_1^2\alpha_1\alpha_2 - \omega_2\alpha_1^2)$$

$$+ D_3(1,1)(\omega_1\omega_2\alpha_2^2 - \omega_1^2\alpha_1\alpha_2)]\cos 2\phi_1$$

$$- \mu[\nu D_1(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + \nu D_1(1,2)(\omega_1\alpha_2\alpha_1^2 - \omega_2\alpha_2^2\alpha_1)$$

$$+ D_2(2,2)(\omega_2\alpha_1^2 - \omega_1\alpha_2\alpha_1) + D_2(1,1)(\omega_1\alpha_2\alpha_1 - \omega_2\alpha_2^2)]$$

$$+ D_3(1,2)(\omega_1\omega_2\alpha_1\alpha_2^2 - \omega_1^2\alpha_2\alpha_1^2) + D_3(2,1)(\omega_1\omega_2\alpha_1 - \omega_1^2\alpha_2)]\sin 2\phi_2$$

$$+ 2\varepsilon[\lambda_2^4(\omega_2\omega_1\alpha_1^2 - \omega_1^2\alpha_1\alpha_2)$$

$$+ \lambda_1^4(\omega_2\omega_1\alpha_2^2 - \omega_1^2\alpha_1\alpha_2)]]\} / [4(\omega_2^2\alpha_1\alpha_2 - \omega_1\omega_2\alpha_2^2 - \omega_1\omega_2\alpha_1^2 + \omega_1^2\alpha_2\alpha_1)]$$

$$\dot{\phi}_1 = \{\mu[\nu D_1(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + \nu D_1(1,2)(\omega_2\alpha_1\alpha_2^2 - \omega_1\alpha_2\alpha_1^2)$$

$$+ D_2(2,2)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_1^2) + D_2(1,1)(\omega_2\alpha_2^2 - \omega_1\alpha_1\alpha_2)$$

$$+ D_3(2,1)(\omega_1^2\alpha_2 - \omega_1\omega_2\alpha_1) + D_3(1,2)(\omega_1^2\alpha_1^2\alpha_2 - \omega_2\omega_1\alpha_2^2)]\cos 2\phi_1$$

$$- \mu[\nu D_1(2,2)(\omega_2\alpha_1^2 - \omega_1\alpha_1\alpha_2) + \nu D_1(1,1)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2^2)$$

$$+ D_2(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + D_2(1,2)(\omega_2\alpha_2^2\alpha_1 - \omega_1\alpha_2\alpha_1^2)$$

$$+ D_3(2,2)(\omega_1^2\alpha_1\alpha_2 - \omega_1\omega_2\alpha_1^2) + D_3(1,1)(\omega_2\omega_1\alpha_2^2 - \omega_1^2\alpha_2\alpha_1)]\ \sin 2\phi_1$$

$$+ Z_2^2\kappa[3C(2,2)^2(\omega_2\alpha_2^2\alpha_1^2 - \omega_1\alpha_2^3\alpha_1) + C(1,1)C(2,2)(\omega_2\alpha_2^4 + \omega_2\alpha_1^2$$

$$- \omega_1\alpha_2^3\alpha_1 - \omega_1\alpha_2\alpha_1) + 3C(1,1)^2(\omega_2\alpha_2^2 - \omega_1\alpha_1\alpha_2)]$$

$$+ \frac{1}{2} Z_1^2\kappa[3C(2,2)^2(\omega_2\alpha_1^4 - \omega_1\alpha_2\alpha_1^3) + C(2,2)C(1,1)(\omega_2\alpha_2^2\alpha_1^2)$$

$$+ \omega_2\alpha_1^2 - \omega_1\alpha_2\alpha_1^3 - \omega_1\alpha_1\alpha_2) + 3C(1,1)^2(\omega_2\alpha_2^2 - \omega_1\alpha_1\alpha_2)]$$

$$+ \lambda[G(\omega_1{}^2\alpha_2\alpha_1{}^2 + \omega_1{}^2\alpha_2$$

$$- \omega_2\omega_1\alpha_2{}^2\alpha_1 - \omega_1\omega_2\alpha_1) + K_{22}(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_1{}^2)$$

$$+ K_{11}(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2{}^2)]]\} \ / \ [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2$$

$$- \omega_2\omega_1\alpha_1^2 + \omega_1^2\alpha_2\alpha_1)]$$

(2) $2\omega_2 = \omega_o$

$$\dot{Z}_2 = Z_2\{\mu[\nu D_1(2,2)(\omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2)$$

$$+ \nu D_1(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1{}^2) + D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1)$$

$$+ D_2(1,2)(\omega_1\alpha_2\alpha_1{}^2 - \omega_2\alpha_2^2\alpha_1) + D_3(2,2)(\omega_2\alpha_1\alpha_2$$

$$- \omega_2\omega_1\alpha_2^2) + D_3(1,1)(\omega_2\omega_1\alpha_1^2 - \omega_2^2\alpha_2\alpha_1)]\cos 2\phi_2$$

$$- \mu[\nu D_1(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + \nu D_1(1,2)(\omega_2\alpha_2^2\alpha_1 - \omega_1\alpha_2\alpha_1^2)$$

$$+ D_2(2,2)(\omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2) + D_2(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1^2)$$

$$+ D_3(2,1)(\omega_1\omega_2\alpha_2 - \omega_2^2\alpha_1) + D_3(1,2)(\omega_1\omega_2\alpha_1^2\alpha_2 - \omega_2^2\alpha_2^2\alpha_1)]\sin 2\phi_2$$

$$+ 2\varepsilon[\lambda_2^4(\omega_2\omega_1\alpha_2^2 - \omega_2^2\alpha_1\alpha_2) + \lambda_1^4(\omega_2\omega_1\alpha_1^2$$

$$- \omega_2^2\alpha_1\alpha_2)]\} \;/\; [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2 - \omega_2\omega_1\alpha_1^2 + \omega_1^2\alpha_1\alpha_2)]$$

$$\dot{\phi}_2 = \{\mu[\nu D_1(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + \nu D_1(1,2)(\omega_1\alpha_2\alpha_1^2 - \omega_2\alpha_2^2\alpha_1)$$

$$+ D_2(2,2)(\omega_2\alpha_1 a_2 - \omega_1\alpha_2^2)$$

$$+ D_2(1,1)(\omega_1\alpha_1^2 - \omega_2\alpha_1\alpha_2) + D_3(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2)$$

$$+ D_3(1,2)(\omega_2^2\alpha_2^2\alpha_1 - \omega_2\omega_1\alpha_2\alpha_1{}^2) \cos 2\phi_2$$

$$- \mu[\nu D_1(2,2)(\omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2) + \nu D_1(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1^2)$$

$$+ D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + D_2(1,2)(\omega_1\alpha_2\alpha_1^2 - \omega_2\alpha_2^2\alpha_1)$$

$$+ D_3(2,2)(\omega_2^2\,\alpha_2\alpha_1 - \omega_2\omega_1\alpha_2^2) + D_3(1,1)(\omega_2\omega_1\alpha_1^2 - \omega_2^2\alpha_2\alpha_1)]\sin 2\phi_2$$

$$+ Z_1^2\,\kappa[3C(2,2)^2(\omega_1\alpha_2^2\alpha_1^2 - \omega_2\alpha_2\alpha_1^3) + C(2,2)C(1,1)(\omega_1\alpha_1^4 + \omega_1\alpha_2^2$$

$$- \omega_2\alpha_1\alpha_2 - \omega_2\alpha_2\alpha_1{}^3) + 3C(1,1)^2(\omega_1\alpha_1{}^2 - \omega_2\alpha_2\alpha_1)$$

$$+ \frac{1}{2}\, Z_2^2\,\kappa[3C(2,2)^2(\omega_1\alpha_2^4 - \omega_2\alpha_2{}^3\alpha_1)$$

$$+ C(2,2)C(1,1)(\omega_1\alpha_2^2 + \omega_1\alpha_2^2\alpha_1^2 - \omega_2\alpha_1\alpha_2 - \omega_2\alpha_2{}^3\alpha_1)$$

$$+ 3C(1,1)^2(\omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2)] + \lambda[G(\omega_2^2\alpha_1\alpha_2^2$$

$$+ \omega_2^2\alpha_1 - \omega_2\omega_1\alpha_2\alpha_1^2 - \omega_2\omega_1\alpha_2) + K_{22}(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^2)$$

$$+ K_{11}(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1^2)]]\} \;/\; [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2$$

$$- \omega_2\omega_1\alpha_1^2 + \omega_1{}^2\alpha_1\alpha_2)]$$

(3) $\quad |\omega_1 + \omega_2| = \omega_0$

$$\dot{Z}_1 = \{Z_2\{\mu[\nu D_1(2,2)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2{}^2) + \nu D_1(1,1)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2)$$

$$
\begin{aligned}
&+ D_2(1,2)(\omega_2\alpha_2^{\;3} - \omega_1\alpha_2^{\;2}\alpha_1) \\
&+ D_2(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + D_3(2,2)(\omega_2\omega_1\alpha_2^2 - \omega_2^2\alpha_2\alpha_1) \\
&+ D_3(1,1)(\omega_2^2\alpha_2^2 - \omega_1\omega_2\alpha_1\alpha_2)] \cos(\phi_1 + \phi_2) \\
&- \mu[\nu D_1(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + \nu D_1(1,2)(\omega_1\alpha_2^2\alpha_1 - \omega_2\alpha_2^{\;3}) \\
&+ D_2(2,2)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^{\;2}) + D_2(1,1)(\omega_1\alpha_2\alpha_1 - \omega_2\alpha_2^{\;2}) \\
&+ D_3(2,1)(\omega_2^2\alpha_1 - \omega_1\omega_2\alpha_2) + D_3(1,2)(\omega_2^{\;2}\alpha_2^{\;3} - \omega_1\omega_2\alpha_2^{\;2}\alpha_1)]\sin(\phi_1 + \phi_2) \\
&+ 2\epsilon Z_1[\lambda_2^4(\omega_1\omega_2\alpha_1^{\;2} - \omega_1^{\;2}\alpha_1\alpha_2) + \lambda_1^4(\omega_2\omega_1\alpha_2^2 \\
&- \omega_1^2\alpha_1\alpha_2)]\} \; / \; [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2 - \omega_1\omega_2\alpha_1^{\;2} + \omega_1^{\;2}\alpha_1\alpha_2)] \\
Z_1\dot{\phi}_1 = \{&Z_2\{\mu[\nu D_1(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + \nu D_1(1,2)(\omega_2\alpha_2^{\;3} \\
&- \omega_1\alpha_2^2\alpha_1) + D_2(2,2)(\omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2) + D_2(1,1)(\omega_2\alpha_2^2 \\
&- \omega_1\alpha_1\alpha_2) + D_3(2,1)(\omega_1\omega_2\alpha_2 - \omega_2^2\alpha_1) + D_3(1,2)(\omega_1\omega_2\alpha_2^2\alpha_1 \\
&- \omega_2^2\alpha_2^{\;3})] \cos(\phi_1 + \phi_2) - \mu[\nu D_1(2,2)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^{\;2}) \\
&+ \nu D_1(1,1)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2^2) + D_2(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) \\
&+ D_2(1,2)(\omega_2\alpha_2^{\;3} - \omega_1\alpha_2^2\alpha_1) + D_3(2,2)(\omega_2\omega_1\alpha_2^2 - \omega_2^2\alpha_1\alpha_2) \\
&+ D_3(1,1)(\omega_2^2\alpha_2^2 - \omega_2\omega_1\alpha_2\alpha_1)]\sin(\phi_1 + \phi_2)\} \\
&+ Z_1\{Z_2^2\kappa[3C(2,2)^2(\omega_2\alpha_2^{\;2}\alpha_1^{\;2} - \omega_1\alpha_2^{\;3}\alpha_1) \\
&+ C(1,1)C(2,2)(\omega_2\alpha_2^{\;4} + \omega_2\alpha_1^2 - \omega_1\alpha_2^{\;3}\alpha_1 - \omega_1\alpha_2\alpha_1) \\
&+ 3C(1,1)^2(\omega_2\alpha_2^2 - \omega_1\alpha_1\alpha_2)] + \frac{1}{2} Z_1^{\;2}\kappa[3C(2,2)^2(\omega_2\alpha_1^{\;4} - \omega_1\alpha_2\alpha_1^{\;3})
\end{aligned}
$$

$$+ C(2,2)C(1,1)(\omega_2\alpha_2^2\alpha_1{}^2 + \omega_2\alpha_1{}^2$$

$$- \omega_1\alpha_2\alpha_1{}^3 - \omega_1\alpha_1\alpha_2) + 3C(1,1)^2(\omega_2\alpha_2{}^2 - \omega_1\alpha_1\alpha_2)]$$

$$+ \lambda[G(\omega_1^2\alpha_2\alpha_1{}^2 + \omega_1{}^2\alpha_2 - \omega_2\omega_1\alpha_2^2\alpha_1 - \omega_1\omega_2\alpha_1)$$

$$+ K_{22}(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_1{}^2) + K_{11}(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2{}^2)]\}/[4(\omega_2{}^2\alpha_1\alpha_2$$

$$- \omega_2\omega_1\alpha_2{}^2 - \omega_2\omega_1\alpha_1{}^2 + \omega_1^2\alpha_1\alpha_2)]$$

$$\dot{Z}_2 = \{Z_1\{\mu[\nu D_1(2,2)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_1{}^2)$$

$$+ \nu D_1(1,1)(\omega_2\alpha_2\alpha_1 - \omega_1\alpha_1{}^2) + D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1)$$

$$+ D_2(1,2)(\omega_1\alpha_1{}^3 - \omega_2\alpha_2\alpha_1{}^2) + D_3(2,2)(\omega_2\omega_1\alpha_1^2 - \omega_1^2\alpha_1\alpha_2)$$

$$+ D_3(1,1)(\omega_1^2\alpha_1^2 - \omega_1\omega_2\alpha_1\alpha_2)]\cos(\phi_1 + \phi_2)$$

$$- \mu[\nu D_1(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + \nu D_1(1,2)(\omega_2\alpha_2\alpha_1^2 - \omega_1\alpha_1{}^3)$$

$$+ D_2(2,2)(\omega_2\alpha_2\alpha_1 - \omega_2\alpha_1{}^2) + D_2(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1{}^2)$$

$$+ D_3(2,1)(\omega_1^2\alpha_2 - \omega_1\omega_2\alpha_1) + D_3(1,2)(\omega_1^2\alpha_1{}^3 - \omega_1\omega_2\alpha_2\alpha_1{}^2)]\sin(\phi_1 + \phi_2)\}$$

$$+ 2\varepsilon Z_2[\lambda_2^4(\omega_2\omega_1\alpha_2{}^2 - \omega_2^2\alpha_1\alpha_2)$$

$$+ \lambda_1^4\,(\omega_2\omega_1\alpha_1^2 - \omega_2^2\alpha_1\alpha_2)]\} \;/\; [4(\omega_2{}^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2{}^2$$

$$- \omega_2\omega_1\alpha_1{}^2 + \omega_1^2\alpha_1\alpha_2)]$$

$$Z_2\dot{\phi}_2 = \{Z_1\{\mu[D_1(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + D_1(1,2)(\omega_1\alpha_1{}^3$$

$$- \omega_2\alpha_2\alpha_1{}^2) + D_2(2,2)(\omega_2\alpha_1^2 - \omega_1\alpha_1\alpha_2)$$

$$+ D_2(1,1)(\omega_1\alpha_1{}^2 - \omega_2\alpha_2\alpha_1) + D_3(2,1)(\omega_2\omega_1\alpha_1 - \omega_1{}^2\alpha_2)$$

$$+ D_3(1,2)(\omega_1\omega_2\alpha_2\alpha_1{}^2 - \omega_1{}^2\alpha_1{}^3)]\cos(\phi_1 + \phi_2)$$

$$- \mu[\nu D_1(2,2)(\omega_1\alpha_2\alpha_1 - \omega_2\alpha_1{}^2) + \nu D_1(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1{}^2)$$

$$+ D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + D_2(1,2)(\omega_1\alpha_1{}^3 - \omega_2\alpha_2\alpha_1{}^2)$$

$$+ D_3(2,2)(\omega_2\omega_1\alpha_1{}^2 - \omega_1^2\alpha_2\alpha_1) + D_3(1,1)(\omega_1{}^2\alpha_1{}^2 - \omega_1\omega_2\alpha_1\alpha_2)]\sin(\phi_1 + \phi_2)\}$$

$$+ Z_2\{Z_1{}^2\kappa[(3C(2,2)^2(\omega_1\alpha_2^2\alpha_1^2 - \omega_2\alpha_2\alpha_1{}^3)$$

$$+ C(2,2)C(1,1)(\omega_1\alpha_1{}^4 + \omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2 - \omega_2\alpha_2\alpha_1{}^3)$$

$$+ 3C(1,1)^2(\omega_1\alpha_1{}^2 - \omega_2\alpha_2\alpha_1) + \frac{1}{2} Z_2^2 \kappa[3C(2,2)^2(\omega_1\alpha_2{}^4 - \omega_2\alpha_2{}^3\alpha_1)$$

$$+ C(2,2)C(1,1)(\omega_1\alpha_2^2 + \omega_1\alpha_2^2\alpha_1{}^2 - \omega_2\alpha_1\alpha_2 - \omega_2\alpha_2{}^3\alpha_1)$$

$$+ 3C(1,1)^2(\omega_1\alpha_1^2 - \omega_2\alpha_1\alpha_2)] + \lambda[G(\omega_2{}^2\alpha_2{}^2\alpha_1$$

$$+ \omega_2{}^2\alpha_1 - \omega_2\omega_1\alpha_2\alpha_1^2 - \omega_2\omega_1\alpha_2) + K_{22}(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^2)$$

$$+ K_{11}(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1^2)]\} \;/\; [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2$$

$$- \omega_2\omega_1\alpha_1{}^2 + \omega_1{}^2\alpha_1\alpha_2)]$$

(4) $|\omega_1 - \omega_2| = \omega_o$

$$\dot{Z}_1 = \{Z_2\{\mu[\nu D_1(2,2)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^2) + \nu D_1(1,1)\ (\omega_2\alpha_2{}^2 - \omega_1\alpha_1\alpha_2)$$

$$+ D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + D_2(1,2)(\omega_2\alpha_2{}^3 - \omega_1\alpha_2^2\alpha_1)$$

$$+ D_3(2,2)(\omega_2{}^2\alpha_1\alpha_2 - \omega_1\omega_2\alpha_2{}^2)$$

$$+ D_3(1,1)(\omega_2^2\alpha_2^2 - \omega_1\omega_2\alpha_1\alpha_2)]\ \cos(\phi_1 - \phi_2)$$

$$+ \mu[\nu D_1(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2) + \nu D_1(1,2)(\omega_1\alpha_2{}^2\alpha_1 - \omega_2\alpha_2{}^3)$$

$$+ D_2(2,2)(\omega_2\alpha_2\alpha_1 - \omega_1\alpha_2^2) + D_2(1,1)(\omega_2\alpha_2^2 - \omega_2\alpha_1\alpha_2)$$

$$+ D_3(2,1)(\omega_2^2\alpha_1 - \omega_2\omega_1\alpha_2) + D_3(1,2)(\omega_2\omega_1\alpha_2{}^2\alpha_1$$

$$- \omega_2{}^2\alpha_2{}^3)]\sin(\phi_1 - \phi_2)\} + 2\varepsilon Z_1(\lambda_2{}^4(\omega_2\omega_1\alpha_1{}^2 - \omega_1{}^2\alpha_1\alpha_2)$$

$$+ \lambda_1(\omega_2\omega_1\alpha_2^2 - \omega_1^2\alpha_1\alpha_2)]\} / [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2$$

$$- \omega_2\omega_1\alpha_1{}^2 + \omega_1^2\alpha_1\alpha_2)]$$

$$Z_1\dot{\phi}_1 = \{Z_2\{\mu[\nu D_1(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1)$$

$$+ \nu D_1(1,2)(\omega_2\alpha_2{}^3 - \omega_1\alpha_2{}^3) + D_2(2,2)(\omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2)$$

$$+ D_2(1,1)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_2^2) + D_3(2,1)(\omega_2\omega_1\alpha_2 - \omega_2^2\alpha_1)$$

$$+ D_3(1,2)(\omega_2^2\alpha_2{}^3 - \omega_1\omega_2\alpha_2{}^2\alpha_1)]\cos(\phi_1 + \phi_2)$$

$$- \mu[\nu D_1(2,2)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^2) + \nu D_1(1,1)(\omega_2\alpha_2^2$$

$$- \omega_1\alpha_1\alpha_2) + D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + D_2(1,2)(\omega_2\alpha_2{}^3$$

$$- \omega_1\alpha_2^2\alpha_1) + D_3(2,2)(\omega_2^2\alpha_1\alpha_2 - \omega_1\omega_2\alpha_2{}^2)$$

$$+ D_3(1,1)(\omega_2^2\alpha_2^2 - \omega_1\omega_2\alpha_1\alpha_2)]\sin(\phi_1 - \phi_2)$$

$$+ Z_1\{Z_2^2\kappa[3C(2,2)^2(\omega_2\alpha_2^2\alpha_1 - \omega_1\alpha_2{}^3\alpha_1)$$

$$+ C(1,1)C(2,2)(\omega_2\alpha_2{}^4 + \omega_2\alpha_1^2 - \omega_1\alpha_2{}^3\alpha_1 - \omega_1\alpha_2\alpha_1)$$

$$+ 3C(1,1)^2(\omega_2\alpha_2^2 - \omega_1\alpha_1\alpha_2)] + \frac{1}{2} Z_1{}^2\kappa[3C(2,2)^2(\omega_2\alpha_1{}^4 - \omega_1\alpha_2\alpha_1{}^3)$$

$$+ C(2,2)C(1,1)(\omega_2\alpha_2^2\alpha_1{}^2 + \omega_2\alpha_1{}^2 - \omega_1\alpha_2\alpha_1{}^3 - \omega_1\alpha_1\alpha_2)$$

$$+ 3C(1,1)^2(\omega_2\alpha_2^2 - \omega_1\alpha_1\alpha_2)]$$

$$+ \lambda[G(\omega_1{}^2\alpha_2\alpha_1{}^2 + \omega_1{}^2\alpha_2 - \omega_2\omega_1\alpha_2^2\alpha_1 - \omega_1\omega_2\alpha_1)$$

$$+ K_{22}(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_1{}^2) + K_{11}(\omega_1\alpha_1\alpha_2$$

$$- \omega_2\alpha_2{}^2)]]\} \;/\; [4(\omega_2^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2{}^2 - \omega_1\omega_2\alpha_1{}^2 + \omega_1{}^2\alpha_1\alpha_2)]$$

$$\dot{Z}_2 = \{Z_1\{\mu(\nu D_1(2,2)(\omega_2\alpha_1{}^2 - \omega_1\alpha_1\alpha_2)$$

$$+ \nu D_1(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1{}^2) + D_2(2,1)(\omega_2\alpha_1 - \omega_1\alpha_2)$$

$$+ D_2(1,2)(\omega_1\alpha_1{}^3 - \omega_2\alpha_2\alpha_1^2) + D_3(2,2)(\omega_1{}^2\alpha_1\alpha_2 - \omega_1\omega_2\alpha_1^2)$$

$$+ D_3(1,1)(\omega_1^2\alpha_1^2 - \omega_1\omega_2\alpha_1\alpha_2)]\cos(\phi_1 - \phi_2) - \mu[\nu D_1(2,1)(\omega_1\alpha_2$$

$$- \omega_2\alpha_1) + D_1(1,2)(\omega_2\alpha_2\alpha_1{}^2\omega_1\alpha_1{}^3) + D_2(2,2)(\omega_2\alpha_1{}^2 - \omega_1\alpha_1\alpha_2)$$

$$+ D_2(1,1)(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1{}^2) + D_3(2,1)(\omega_2\omega_1\alpha_1 - \omega_1{}^2\alpha_2)$$

$$+ D_3(1,2)(\omega_1{}^2\alpha_1{}^3 - \omega_1\omega_2\alpha_2\alpha_1{}^2)]\sin(\phi_1 - \phi_2)\} + 2\varepsilon Z_2[\lambda_2^4(\omega_2\omega_1\alpha_2{}^2$$

$$- \omega_2{}^2\alpha_1\alpha_2) + \lambda_1{}^4(\omega_2\omega_1\alpha_1{}^2 - \omega_2{}^2\alpha_1\alpha_2)] \;/\; [4(\omega_2{}^2\alpha_1\alpha_2$$

$$- \omega_2\omega_1\alpha_2{}^2 - \omega_1\omega_2\alpha_1{}^2 + \omega_1{}^2\alpha_1\alpha_2)]$$

$$Z_2\dot{\phi}_2 = \{Z_1\{\mu[\nu D_1(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + \nu D_1(1,2)(\omega_2\alpha_2\alpha_1{}^2 - \omega_1\alpha_1{}^3)$$

$$+ D_2(2,2)(\omega_2\alpha_1^2 - \omega_1\alpha_2\alpha_1) + D_2(1,1)(\omega_2 a_2\alpha_1 - \omega_1\alpha_1{}^2)$$

$$+ D_3(2,1)(\omega_2\omega_1\alpha_1 - \omega_1{}^2\alpha_2) + D_3(1,2)(\omega_1{}^2\alpha_1{}^3 - \omega_2\omega_1\alpha_2\alpha_1{}^2)]\cos(\phi_1 - \phi_2)$$

$$- \mu[\nu D_1(2,2)(\omega_1\alpha_1\alpha_2 - \omega_2\alpha_1{}^2) + \nu D_1(1,1)(\omega_1\alpha_1^2 - \omega_2\alpha_1\alpha_2)$$

$$+ D_2(2,1)(\omega_1\alpha_2 - \omega_2\alpha_1) + D_2(1,2)(\omega_2\alpha_2\alpha_1{}^2 - \omega_1\alpha_1{}^3)$$

$$+ D_3(2,2)(\omega_2\omega_1\alpha_1^2 - \omega_1{}^2\alpha_2\alpha_1) + D_3(1,1)(\omega_2\omega_1\alpha_2\alpha_1 - \omega_1{}^2\alpha_1{}^2)]\sin(\phi_1 - \phi_2)$$

$$+ Z_2\{Z_1{}^2\kappa[3C(2,2)^2(\omega_1\alpha_2^2\alpha_1^2 - \omega_2\alpha_2\alpha_1{}^3)$$

$$+ C(2,2)C(1,1)(\omega_1\alpha_1{}^4 + \omega_1\alpha_2^2 - \omega_2\alpha_1\alpha_2 - \omega_2\alpha_2\alpha_1{}^3)$$

$$+ 3C(1,1)^2(\omega_1\alpha_1{}^2 - \omega_2\alpha_2\alpha_1) + \frac{1}{2} Z_2^2 \kappa[3C(2,2)^2(\omega_1\alpha_2{}^4 - \omega_2\alpha_2{}^3\alpha_1)$$

$$+ C(2,2)C(1,1)(\omega_1\alpha_2^2 + \omega_1\alpha_2^2 \alpha_1{}^2 - \omega_2\alpha_1\alpha_2 - \omega_2\alpha_2{}^3\alpha_1)$$

$$+ 3C(1,1)^2(\omega_1\alpha_1{}^2 - \omega_2\alpha_1\alpha_2)] + \lambda[G(\omega_2{}^2\alpha_2{}^2\alpha_1$$

$$+ \omega_2{}^2\alpha_1 - \omega_2\omega_1\alpha_2\alpha_1^2 - \omega_2\omega_1\alpha_2) + K_{22}(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_2^2)$$

$$+ K_{11}(\omega_2\alpha_1\alpha_2 - \omega_1\alpha_1{}^2)]]\} \;/\; [4(\omega_2{}^2\alpha_1\alpha_2 - \omega_2\omega_1\alpha_2^2$$

$$- \omega_1\omega_2\alpha_1{}^2 + \omega_1{}^2\alpha_2\alpha_1)]$$

APPENDIX F2

DEFINITION OF PARAMETERS IN THE AVERAGED EQUATIONS

I. The parameters in the averaged equations are defined as follows:

$$D_i = \{D_{i,rs}\}_{2x2} ,$$

$$A_j^k = [\nu D_{1,k1} - (D_{3,k2}\alpha_j + D_{4,k1}\omega_j)] ,$$

$$B_j^k = [\nu D_{1,k2}\,\alpha_j + (D_{3,k1} - D_{4,k2}\alpha_j\omega_j)] ,$$

$$U_{11} = (\Delta_2 A_1^2 + \Delta_1\alpha_2 B_1^1) / (4\Delta) ,$$

$$U_{22} = (-\,\Delta_2 A_2^2 - \Delta_1\alpha_1 B_2^1) / (4\Delta) ,$$

$$V_{11} = (\Delta_2 B_1^2 - \Delta_1\alpha_2 A_1^1) / (4\Delta) ,$$

$$V_{22} = (-\,\Delta_2 B_2^2 + \Delta_1\alpha_1 A_2^1) / (4\Delta) ,$$

$$U_{12} = (\Delta_2 A_2^2 + \Delta_1\alpha_2 B_2^1) / (4\Delta) ,$$

$$U_{21} = (-\,\Delta_2 A_1^2 - \Delta_1\alpha_1 B_1^1) / (4\Delta) ,$$

$$V_{12} = (\Delta_2 B_2^2 - \Delta_1\alpha_2 A_2^1) / (4\Delta) ,$$

$$V_{21} = (-\,\Delta_2 B_1^2 + \Delta_1\alpha_1 A_1^1) / (4\Delta) ,$$

$$\hat{U}_{12} = (\Delta_2 A_2^2 - \Delta_1\alpha_2 B_2^1) / (4\Delta) ,$$

$$\hat{U}_{21} = (\Delta_2 A_1^2 - \Delta_1\alpha_1 B_1^1) / (4\Delta) ,$$

$$\hat{V}_{12} = (\Delta_2 B_2^2 + \Delta_1\alpha_2 A_2^1) / (4\Delta) ,$$

$$\hat{V}_{21} = (\Delta_2 B_1^2 + \Delta_1\alpha_1 A_1^1) / (4\Delta) ,$$

where $\Delta < 0$.

II. In the averaged equations the damping, nonlinearity and detuning terms are defined as follow: $(r = 1,2)$

1. Damping terms:

$$\xi_r = \{2Z_r[\lambda_2^4 \, \omega_r \alpha_r \Delta_2 + \lambda_1^4 \omega_r \alpha_{3-r} \Delta_1] \, (-1)^{r+1}\} \, / \, (4\Delta) \; ,$$

$$\hat{\xi}_r = E^* \xi_r \; .$$

2. Nonlinearity terms:

$$\bar{N}_r^1 = (\kappa/2)[(3C_{22}{}^2 \alpha_r{}^3 + C_{22}C_{11}\alpha_r)\Delta_2 + (C_{22}C_{11}\alpha_r{}^2 \alpha_{3-r} + 3C_{11}{}^2 \alpha_{3-r})\Delta_1],$$

$$\bar{N}_r^2 = (\kappa/2)[(3C_{22}{}^2 \alpha_r \alpha_{3-r}^2 + C_{11}C_{22}\alpha_r)\Delta_2 + (C_{22}C_{11}\alpha_{3-r}^3 + 3C_{11}{}^2 \alpha_{3-r})\Delta_1],$$

$$N_r = [(\bar{N}_r^1 Z_r{}^2 + 2\bar{N}_r^2 Z_{3-r}{}^2) \, (-1)^r] \, / \, (4\Delta) \; ,$$

and let

$$N_1 = N_{11} Z_1{}^2 + 2N_{12} Z_2{}^2 \; ,$$

$$N_2 = 2N_{21} Z_1{}^2 + N_{22} Z_2{}^2 \; .$$

3. Detuning terms:

$$\lambda D_r = \lambda\{4[G(\omega_{3-r}\omega_r\alpha_r + \omega_{3-r}\omega_r\alpha_{3-r}^2\alpha_r - \omega_r^2\alpha_{3-r}\alpha_r^2 - \omega_r{}^2\alpha_{3-r})$$

$$+ K_{22}(\omega_{3-r}{}^2\alpha_r{}^2 - \omega_r\alpha_r\alpha_{3-r}) + K_{11}(\omega_{3-r}\alpha_{3-r}{}^2 - \omega_r\alpha_r\alpha_{3-r})]\}/(4\Delta).$$

APPENDIX F3

STABILITY CRITERIA OF TRIVIAL SOLUTION AND FORMULATION OF LOCAL BIFURCATION ANALYSIS

The stability of the trivial solution is determined by the eigenvalues of the linear part of Eq. (4.3), i.e.,

$$\Omega^4 + a_3\Omega^3 + a_2\Omega^2 + a_1\Omega + a_o = 0 \qquad (C1)$$

where

$$a_3 = 2\,E^*(\xi_1 + \xi_2) \geq 0\ ,$$

$$a_2 = \lambda^2(D_1^2 + D_2^2) + E^{*2}(\xi_1 + \xi_2)^2 - 2\eta\ ,$$

$$a_1 = 2\,E^*\lambda^2(D_1^2\xi_2 + D_2^2\xi_1) - 2\,E^*(\xi_1 + \xi_2)\,\eta\ ,$$

$$a_o = (\eta - \lambda^2 D_1 D_2)^2 + \lambda^2 E^{*2}(D_1\xi_2 - D_2\xi_1)^2 \geq 0\ ,$$

and

$$\eta = (h^2 R_p - E^{*2}\xi_1\xi_2)\ .$$

The stability criterion can be written

$$a_o > 0\ ,\ a_1 > 0\ ,\ a_2 > 0\ ,\ a_3 > 0 \ \text{ and}$$

$$a_4 = a_1a_2a_3 - a_oa_3^2 - a_1^2$$

$$= 4E^{*2}[\lambda^2\xi_1\xi_2(D_1 + D_2)^2 - (\xi_1 + \xi_2)^2\eta]\cdot[\lambda^2(D_1 - D_2)^2 + E^{*2}(\xi_1 + \xi_2)^2] > 0.$$

It is obvious that for $R_p < 0$, i.e., $\eta < 0$, the trivial solution is always stable and for $R_p > 0$, the stability criterion reduces to

$$(\xi_1 + \xi_2) > 0\ ,\ \lambda^2\xi_1\xi_2(D_1 + D_2)^2 - (\xi_1 + \xi_2)^2\eta > 0\ . \qquad (C2)$$

The trivial solution loses stability when

$$\lambda_c^{\pm} = \pm \left[\left(\frac{\xi_1}{\xi_2}\right)^{1/2} + \left(\frac{\xi_2}{\xi_1}\right)^{1/2} \right] n^{1/2} / (D_1 + D_2) \, , \tag{C3}$$

and the eigenvalues are

$$\Omega_{1,2} = \pm i \left| \left(\frac{\xi_1}{\xi_2}\right)^{1/2} D_2 - \left(\frac{\xi_2}{\xi_1}\right)^{1/2} D_1 \right| n^{1/2} / (D_1 + D_2) \, ,$$

and

$$\Omega_{3,4} = - \left(\frac{a_3}{2}\right) \pm \left[\left(\frac{a_3}{2}\right)^2 - \frac{a_o a_3}{a_1} \right]^{1/2} . \tag{C4}$$

From the direct and adjoint eigenvalues problems, i.e., $A\underset{\sim}{a} = \Omega \underset{\sim}{a}$ with $\underset{\sim}{a} = \underset{\sim}{c} + i\underset{\sim}{d}$ and $A^T\underset{\sim}{b} = \Omega\underset{\sim}{b}$ with $\underset{\sim}{b} = \underset{\sim}{e} + i\underset{\sim}{f}$, one can obtain the eigenvectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, respectively at $\lambda = \lambda_c^{\pm}$, at λ_c^{+} and λ_c^{-} the eigenvectors may be different even though Ω_i are the same. Furthermore, depending on whether $\Omega_{3,4}$ are complex or real eigenvalues one can make use of the transformation

$$X_j = 2(c_j^1 u_1 + d_j^1 u_2) + 2(c_j^2 v_1 + d_j^2 v_2) \, ,$$

$$Y_j = 2(c_{2+j}^1 u_1 + d_{2+j}^1 u_2) + 2(c_{2+j}^2 v_1 + d_{2+j}^2 v_2) \, .$$

or

$$X_j = 2\,(c_j^1 u_1 + d_j^1 u_2) + a_j^3 v_1 + a_j^4 v_2 \, , \tag{C5}$$

$$Y_j = 2(c_{j+2}^1 u_1 + d_{2+j}^1 u_2) + a_{2+j}^3 v_1 + a_{2+j}^4 v_2 \, ,$$

to reduce the linear operator to

$$\mathrm{diag}\left\{\begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \begin{bmatrix} -\delta_2 & \omega_2 \\ -\omega_2 & -\delta_2 \end{bmatrix}\right\} \text{ or } \mathrm{diag}\left\{\begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \lambda_3, \lambda_4\right\} . \quad (C6)$$

Superscripts here, denote the eigenvectors corresponding to the critical eigenvalue. The δ's and ω's in Eq. (C4) are the real and the imaginary parts of the eigenvalue. It is evident that the averaged system exhibits a Hopf bifurcation at $\lambda = \lambda_c$ and the results of the bifurcating solutions are given in Eq. (4.21). In order to obtain the values of R and S given by Eq. (4.22), it is sufficient to express g_1 and g_2 in terms of new variables u_1 and u_2 alone. This is due to the fact that v_1 and v_2 can be neglected in the first approximation, since the nonlinearities are cubic in $\underline{u}$ and $\underline{v}$. Thus putting

$$
\begin{aligned}
F_1 &= -\left[N_{11}(X_1^2 + Y_1^2) + 2N_{12}(X_2^2 + Y_2^2)\right]Y_1 , \\
F_2 &= -\left[2N_{21}(X_1{}^2 + Y_1{}^2) + N_{22}(X_2{}^2 + Y_2{}^2)\right]Y_2 , \\
F_3 &= \left[N_{11}(X_1{}^2 + Y_1{}^2) + 2N_{12}(X_2{}^2 + Y_2{}^2)\right]X_1 , \\
F_4 &= \left[2N_{21}(X_1{}^2 + Y_1{}^2) + N_{22}(X_2{}^2 + Y_2{}^2)\right] X_2 ,
\end{aligned}
\quad (C7)
$$

where $X_j = 2(c_j^1 u_1 + d_j^1 u_2)$, $\quad Y_j = 2(c_{2+j}^1 u_1 + d_{2+j}^1 u_2)$

and

$$S = \begin{bmatrix} e_1^1 & e_2^1 & e_3^1 & e_4^1 \\ -f_1^1 & -f_2^1 & -f_3^1 & -f_4^1 \end{bmatrix} \quad (C8)$$

we can write $\underset{\sim}{G} = \underset{\sim}{S} \cdot \underset{\sim}{F}$ as functions of u_1 and u_2.

APPENDIX F4

STABILITY CRITERIA OF NONTRIVIAL SOLUTION

The stability of the characteristic equation (4.12) is given by the Routh-Hurwitz criteria, i.e., for equation

$$p^3 + a_2 p^2 + a_1 p + a_0 = 0 ,$$

where $a_2 = 2E^*(\xi_1 + \xi_2)$,

$$a_1 = E^*(\xi_1 + \xi_2)^2 + F^2\alpha \mp \alpha^{1/2} [\sigma_1 Z_{10}(\xi_1/\xi_2)^{1/2}$$

$$+ \sigma_2 Z_{20}(\xi_2/\xi_1)^{1/2}] ,$$

$$a_0 = \mp 2E^*(\xi_1/\xi_2)^{1/2}\alpha^{1/2}[Z_{10}\sigma_1 + Z_{20}\sigma_2] .$$

The stability criterion can be written as follow:

$$a_0 > 0 , \quad a_1 > 0 , \quad a_2 > 0 \quad \text{and}$$

$$a_1 a_2 - a_0 = 2E^{*3}(\xi_1 + \xi_2)^3 + 2F^2\alpha E^*(\xi_1 + \xi_2)$$

$$\mp 2E^*\alpha^{1/2}\{(\xi_1 + \xi_2)[\sigma_1 Z_{10}(\xi_1/\xi_2)^{1/2} + \sigma_2 Z_{20}(\xi_2/\xi_1)^{1/2}]$$

$$- (\xi_1\xi_2)^{1/2}[Z_{10}\sigma_1 + Z_{20}\sigma_2]\} > 0$$

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